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Entropic uncertainty relations for mutually unbiased operator frames

Jesni Shamsul Shaari^{1,2} and Stefano Mancini^{3,4,*}

¹ Department of Physics, Faculty of Science, International Islamic University Malaysia (IIUM), Jalan Sultan Ahmad Shah, Bandar Indera Mahkota, 25200 Kuantan, Pahang, Malaysia

² IIUM Photonics and Quantum Centre, International Islamic University Malaysia (IIUM), Jalan Sultan Ahmad Shah, Bandar Indera Mahkota, 25200 Kuantan, Pahang, Malaysia

³ School of Science & Technology, University of Camerino, I-62032 Camerino, Italy

⁴ INFN Sezione di Perugia, I-06123 Perugia, Italy

* Author to whom any correspondence should be addressed.

E-mail: stefano.mancini@unicam.it and jesni@iium.edu.my

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Abstract

We develop an operator-frame formulation of entropic uncertainty relations in the Hilbert–Schmidt space of operators. For general continuous indexed operator frames, we derive an entropic uncertainty relation for the associated coefficient distributions by combining endpoint norm estimates with Riesz–Thorin interpolation. We then identify a distinguished class of mutually unbiased operator frames, defined through constant-modulus trace overlaps. Under suitable structural conditions, the corresponding coefficient amplitudes are related by a bilinear Fourier transform, leading to a stronger Hirschman–Beckner-type entropic uncertainty relation. As canonical realizations, we consider Weyl displacement operators and Wigner kernels, as well as Cartesian dyadic frames generated by position and momentum eigenstates. These examples recover familiar continuous-variable Fourier dualities while extending entropic uncertainty relations beyond measurement outcomes to operator representations themselves.

1. Introduction

The uncertainty principle occupies a central role in quantum theory, originating from Heisenberg’s observation that complementary physical quantities such as position and momentum cannot be simultaneously specified with arbitrary precision [1]. In its standard formulation, this complementarity arises from the Fourier relationship between the position-space and momentum-space wavefunction amplitudes. More precisely, a quantum state whose wavefunction is sharply localized in position space necessarily possesses a broadly distributed momentum-space wavefunction, and vice versa. The resulting uncertainty relations may be expressed either in terms of variances, as in the Kennard–Robertson formulation [2], or, more generally, through entropic measures associated with the corresponding probability distributions [3–5]. An extensive review of entropic uncertainty relations and their applications may be found in [6]. In continuous-variable systems, entropic uncertainty relations have been extensively developed beyond the canonical position–momentum setting, including formulations for arbitrary quadratures, multimode systems, and (unitary) linear canonical transformations [7, 8]. Uncertainty relations have also been formulated in the framework of countable frame representations [9]. In finite-dimensional systems, entropic uncertainty relations were later formulated in terms of overlaps between orthonormal bases by Maassen and Uffink [10]. Mutually unbiased bases (MUBs) [11, 12] correspond to the maximally complementary case, yielding the strongest entropic lower bounds. Simply put, two orthonormal bases are said to be mutually unbiased when complete knowledge of a state in one basis implies maximal uncertainty in the other. In finite-dimensional quantum systems, MUBs have found important applications in quantum information, quantum tomography, and quantum cryptography. The

notion of mutual unbiasedness may also be extended to continuous-variable systems, where the position and momentum eigenbases provide a canonical example [13].

The idea of complementarity has also been extended from state vectors to operator spaces through the notion of mutually unbiased unitary bases, where unitary operators form orthogonal bases with constant Hilbert–Schmidt overlaps [14, 15]. In this setting, maximal entropic uncertainty between operator measurements is likewise associated with mutually unbiased operator structures, closely paralleling the role played by MUBs for ordinary observables [16, 17].

Motivated by these developments, in the present work we formulate a general operator-frame approach for continuous-variable systems. Beginning with families of operators acting on a Hilbert space, we derive a general entropic uncertainty relation by establishing suitable endpoint norm estimates and applying the Riesz–Thorin interpolation theorem (see theorem IX.17 of [18]). We then introduce a notion of mutual unbiasedness for operator families, defined through constant-modulus trace overlaps between operator pairings. Under additional conditions ensuring a sufficiently regular overlap structure, this mutual unbiasedness condition induces a generalized Fourier transform relation between the corresponding operator coefficient functions, yielding an explicit realization of the general uncertainty framework. Unlike conventional continuous-variable entropic uncertainty relations, which concern probability distributions associated with measurements of observables, the present formulation is based on coefficient distributions arising from operator expansions. The resulting uncertainty relation expresses the impossibility of simultaneously constructing operator representations that are sharply localized in mutually unbiased operator frames. While the initial derivation shares certain interpolation-theoretic ingredients with frame-based uncertainty relations [9], the present framework is formulated in terms of continuous operator frames and culminates in an operator-level notion of mutual unbiasedness. More precisely, the continuous parameterization of the operator families allows us to consider continuous translational covariance in section 3 together with the constant-modulus overlap condition, it leads to the bilinear phase structure and the resulting Fourier-dual relations. This then enables the derivation of Fourier-type entropic uncertainty relations, yielding stronger lower bounds than those obtained in the general operator-frame setting.

This continuous structure also allows us to specialize the general framework to two physically relevant continuous-variable settings. First, choosing an antisymmetric bilinear form on phase space yields the symplectic Fourier structure associated with displacement operators and Wigner kernels [19–21]. Second, choosing a symmetric bilinear form in Cartesian space leads to a dyadic operator representation associated with position and momentum variables, for which the resulting operator-kernel Fourier duality naturally connects to the ordinary Heisenberg position–momentum uncertainty principle. In this sense, the present framework extends the usual notion of Fourier complementarity from wavefunction amplitudes to mutually unbiased operator representations in Hilbert–Schmidt space.

With this motivation in place, we now develop the general operator-frame framework from which the resulting Fourier dualities and entropic uncertainty relations emerge.

2. Continuous operator frames and Hilbert–Schmidt representations

Let $\mathcal{H} = L^2(\mathbb{R})$ be the space of square-integrable functions on $\mathbb{R}$. Throughout the paper, given the continuous-variable setting, the label space may equivalently be viewed as $\mathbb{C}$ or $\mathbb{R}^2$, since $\mathbb{C} \cong \mathbb{R}^2$ as real vector spaces. While the complex notation is natural for operator formulations in quantum mechanics, we shall frequently use the equivalent real two-dimensional representation when discussing bilinear kernels and symplectic structures.

We consider continuous families of operators $\{O(\lambda)\}$ and $\{P(\beta)\}$ with $\lambda, \beta \in \mathbb{R}^2$ forming generalized orthogonal continuous operator frames, satisfying

$$\mathrm{Tr}[O^\dagger(\lambda)O(\lambda')] = C_O\delta^{(2)}(\lambda - \lambda'), \quad \mathrm{Tr}[P^\dagger(\beta)P(\beta')] = C_P\delta^{(2)}(\beta - \beta') \quad (1)$$

for some constant $C_O, C_P > 0$. We shall study representations of Hilbert–Schmidt operators $A \in B_2(\mathcal{H})$, where $B_2(\mathcal{H})$ is the space of Hilbert–Schmidt operators on the Hilbert space $\mathcal{H}$. This space is equipped with the inner product

$$\langle A, B \rangle := \mathrm{Tr}(A^\dagger B), \quad (2)$$

which induces the norm $\|A\|_{\mathrm{HS}}^2 = \mathrm{Tr}(A^\dagger A)$. Any such operator A then admits the expansion

$$A = \frac{1}{C_O} \int_{\mathbb{R}^2} a_O(\lambda) O(\lambda) d^2\lambda = \frac{1}{C_P} \int_{\mathbb{R}^2} a_P(\beta) P(\beta) d^2\beta \quad (3)$$

where the coefficient function $a_O(\lambda), a_P(\beta) \in L^2(\mathbb{R}^2)$ are given by

$$a_O(\lambda) = \text{Tr}[O^\dagger(\lambda)A], \quad a_P(\beta) = \text{Tr}[P^\dagger(\beta)A]. \quad (4)$$

The coefficient functions $a_O(\lambda)$ and $a_P(\beta)$ constitute two alternative representations of the same Hilbert–Schmidt operator. To quantify the extent to which these representations may be simultaneously localized, we seek an entropic uncertainty relation between the corresponding coefficient distributions. We begin by introducing the normalized coefficient amplitudes $f_O(\lambda)$ and $f_P(\beta)$ as

$$f_O(\lambda) = \frac{a_O(\lambda)}{\sqrt{C_O}\|A\|_{\text{HS}}}, \quad f_P(\beta) = \frac{a_P(\beta)}{\sqrt{C_P}\|A\|_{\text{HS}}}, \quad (5)$$

so that the corresponding coefficient densities, $|f_O(\lambda)|^2$ and $|f_P(\beta)|^2$ are normalized,

$$\int_{\mathbb{R}^2} |f_O(\lambda)|^2 d^2\lambda = \int_{\mathbb{R}^2} |f_P(\beta)|^2 d^2\beta = 1. \quad (6)$$

The two normalized coefficient functions are related by the transformation induced by the change of operator frame. Let us define an operator T , such that $f_P = Tf_O$. Making use of equations (3) and (4) we write,

$$f_P(\beta) = (Tf_O)(\beta) = \int_{\mathbb{R}^2} K(\beta, \lambda) f_O(\lambda) d^2\lambda, \quad (7)$$

with normalized kernel

$$K(\beta, \lambda) = \frac{\text{Tr}[P^\dagger(\beta)O(\lambda)]}{\sqrt{C_OC_P}}. \quad (8)$$

For a well-defined and uniformly bounded $K(\beta, \lambda)$, i.e., $\sup_{\beta, \lambda \in \mathbb{R}^2} |K(\beta, \lambda)| < \infty$, we commit to an interpolation framework not unlike that used in deriving Hirschman’s entropy inequality for the Fourier transform, but carried out in the more general setting of operator-frame transforms. We establish $L^1 \rightarrow L^\infty$ and $L^2 \rightarrow L^2$ bounds for the transform T , interpolate these bounds via the Riesz–Thorin theorem [18] to obtain a Hausdorff–Young-type inequality, and subsequently differentiate the resulting logarithmic norm inequality at $p = q = 2$ to derive a Shannon entropy relation. The details of this is provided in the [appendix](#).

The resulting entropic inequality reads

$$H_O + H_P \geq -2 \ln \left[\sup_{\beta, \lambda} |K(\beta, \lambda)| \right], \quad (9)$$

where H_O and H_P denote the Shannon differential entropy (when finite), given by

$$H_O := - \int_{\mathbb{R}^2} |f_O(\lambda)|^2 \ln |f_O(\lambda)|^2 d^2\lambda, \quad H_P := - \int_{\mathbb{R}^2} |f_P(\beta)|^2 \ln |f_P(\beta)|^2 d^2\beta. \quad (10)$$

The inequality therefore expresses an operator-level entropic uncertainty principle: localization of an operator in one operator basis necessarily implies delocalization in the other. In particular, no Hilbert–Schmidt operator can possess arbitrarily sharp coefficient distributions simultaneously in both operator representations. In this sense, the uncertainty principle emerges not at the level of observables or quantum states alone, but intrinsically at the level of operator representations themselves and is determined by the overlap of the operators.

The operator frames also admit a natural tomographic interpretation. Quite generally, each continuous operator frame may be operationally viewed as defining a distinct measurement or probing scheme for operators, corresponding to a distinct experimental setup, with the expansion coefficients, e.g. $a_O(\lambda)$, providing the corresponding tomographic components for reconstruction. The normalized coefficient distributions, e.g. $|f_O(\lambda)|^2$, characterize how the Hilbert–Schmidt content of A is distributed across the representation. Thus, two frames provide incompatible operational perspectives on the same physical object, and the entropic uncertainty relation quantifies the corresponding minimal uncertainty, or noise floor, for jointly accessing both descriptions. The specific operational realization depends on the physical nature of A and on the choice of operator frames.

For a quantum state, where $A = \rho$ is a density operator, the coefficient functions associated with a given operator frame provide an alternative tomographic representation of the state. In particular, the

choice of frame determines how the state may be probed and reconstructed, while the corresponding normalized coefficient distributions quantify how the state is distributed over the chosen operator representation. More generally, the same framework may be applied to quantum processes through an appropriate operator representation, such as a Choi representation, whenever the resulting operator belongs to the relevant Hilbert–Schmidt space and the continuous family of operators used for its representation forms a suitable operator frame. In this case, the corresponding coefficient functions provide a process-tomographic representation, with the entropic uncertainty relation quantifying the trade-off between complementary process representations.

3. Mutually unbiased operator frames

We now show that when the operator families satisfy a mutual unbiasedness condition, the associated transform acquires the structure of a generalized Fourier transform. This additional structure permits the use of Fourier-analytic techniques and leads to a distinguished class of entropic uncertainty relations.

Let us assume that the operator families $\{O(\lambda)\}$ and $\{P(\beta)\}$ are mutually unbiased in the sense that their trace overlaps, whenever well-defined, satisfy,

$$|\mathrm{Tr}[P^\dagger(\beta)O(\lambda)]| = C, \quad \forall \lambda, \beta \in \mathbb{R}^2. \quad (11)$$

This implies that the overlap can be written as

$$\mathrm{Tr}[P^\dagger(\beta)O(\lambda)] = Ce^{-i\phi(\beta,\lambda)}, \quad (12)$$

for some real-valued phase function $\phi(\beta, \lambda)$. The trace overlap appearing in equation (11) admits a natural interpretation in terms of the action of the operator families on the underlying Hilbert space $\mathcal{H} = L^2(\mathbb{R})$. Recall that for an orthonormal basis $\{e_n\} \subset \mathcal{H}$, we can write

$$\mathrm{Tr}[P^\dagger(\beta)O(\lambda)] = \sum_n \langle P(\beta)e_n, O(\lambda)e_n \rangle. \quad (13)$$

Likewise, in cases involving a continuous resolution of the identity on $\mathcal{H}$,

$$I = \int |\xi\rangle\langle\xi| d\xi,$$

one formally has

$$\mathrm{Tr}[P^\dagger(\beta)O(\lambda)] = \int \langle\xi|P^\dagger(\beta)O(\lambda)|\xi\rangle d\xi. \quad (14)$$

The trace pairing may therefore be understood as the total overlap between the transformed functions generated by the two operator families, summed or integrated over a complete resolution of the underlying Hilbert space. In this sense, the mutual unbiasedness condition expresses the fact that the absolute value of this overlap possesses a constant magnitude independent of λ and β . This reflects the idea that no element of one operator family is preferentially aligned with any element of the other.

As the parameters λ and β serve as labels, or coordinates, on the homogeneous space $\mathbb{R}^2$, no point in this space is distinguished a priori. It is therefore natural to require that the relation between the operator families respects this homogeneity. Accordingly, we impose covariance under translations of the labels, namely that for any $\lambda \in \mathbb{R}^2$, a shift $\lambda \mapsto \lambda + \lambda_0$ modifies the overlap only by a phase factor,

$$\mathrm{Tr}[P^\dagger(\beta)O(\lambda + \lambda_0)] = e^{-i\chi_\beta(\lambda_0)} \mathrm{Tr}[P^\dagger(\beta)O(\lambda)], \quad (15)$$

with an analogous condition for translations in β . We should emphasize that this choice is not claimed to be unique; rather, it is the simplest symmetry-compatible assumption ensuring that the relation between the operator families depends only on relative structure in the label space. Assuming sufficient regularity of the phase function $\phi(\beta, \lambda)$, we have the following proposition.

Proposition 1. Let $\{O(\lambda)\}$ and $\{P(\beta)\}$ with $\lambda, \beta \in \mathbb{R}^2$ be two continuous operator families satisfying

$$\text{Tr}[P^\dagger(\beta)O(\lambda)] = Ce^{-i\phi(\beta, \lambda)}, \quad C > 0.$$

Suppose that the phase function $\phi(\beta, \lambda)$ is sufficiently regular (for instance continuous in each label) and furthermore that their overlap is covariant under translations of the labels, in the sense that

$$\text{Tr}[P^\dagger(\beta)O(\lambda + \lambda_0)] = e^{-i\chi_\beta(\lambda_0)} \text{Tr}[P^\dagger(\beta)O(\lambda)], \quad (16)$$

$$\text{Tr}[P^\dagger(\beta + \beta_0)O(\lambda)] = e^{-i\eta_\lambda(\beta_0)} \text{Tr}[P^\dagger(\beta)O(\lambda)], \quad (17)$$

where $\chi_\beta(\lambda_0)$ is independent of λ and $\eta_\lambda(\beta_0)$ is independent of β . Then the overlap necessarily assumes the form

$$\text{Tr}[P^\dagger(\beta)O(\lambda)] = Ce^{-iB(\beta, \lambda)},$$

where $B : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is bilinear.

Proof. Given $\text{Tr}[P^\dagger(\beta)O(\lambda)] = Ce^{-i\phi(\beta, \lambda)}$ for some real-valued phase function $\phi(\beta, \lambda)$, the first covariance condition gives

$$Ce^{-i\phi(\beta, \lambda + \lambda_0)} = e^{-i\chi_\beta(\lambda_0)} Ce^{-i\phi(\beta, \lambda)}. \quad (18)$$

Hence, modulo integer multiples of 2π ,

$$\phi(\beta, \lambda + \lambda_0) - \phi(\beta, \lambda) = \chi_\beta(\lambda_0). \quad (19)$$

The right-hand side is independent of λ , so for fixed β the increment of ϕ under translations of λ depends only on the translation. Therefore, under mild regularity assumptions such as continuity, ϕ is affine in λ . Applying the second covariance condition similarly implies that ϕ is affine in β . Up to phase factors depending only on one label, which may be absorbed into redefinitions of $O(\lambda)$ and $P(\beta)$, the remaining phase is bilinear:

$$\phi(\beta, \lambda) = B(\beta, \lambda). \quad (20)$$

Thus

$$\text{Tr}[P^\dagger(\beta)O(\lambda)] = Ce^{-iB(\beta, \lambda)}. \quad (21)$$

□

We shall assume throughout that the bilinear pairing B , which emerges from compatibility between mutual unbiasedness and translational covariance on the label space also is non-degenerate. This assumption is motivated by the observation that, if B were degenerate, there would exist nonzero label directions along which the phase factor $e^{-iB(\beta, \lambda)}$ becomes independent of the complementary label variable. The overlap kernel would then become insensitive to variations along such directions, leading to a partially trivial overlap structure despite the constant-modulus condition. The non-degeneracy condition therefore ensures that the pairing B identifies the β - and λ -label spaces as dual variables: no nonzero β pairs trivially with all λ , and no nonzero λ pairs trivially with all β . Accordingly, we write

$$B(\beta, \lambda) = \beta^T M \lambda, \quad \det M \neq 0, \quad (22)$$

where M is an invertible real 2×2 matrix. We shall make use of this equation explicitly later. Substituting the mutual unbiasedness relation (21) into the kernel definition of equation (8), we obtain

$$K(\beta, \lambda) = \frac{C}{\sqrt{C_O C_P}} e^{-iB(\beta, \lambda)}. \quad (23)$$

We shall continue to denote by f_O and f_P the normalized coefficient amplitudes defined in equation (5) and the transform (7) relating these amplitudes now assumes the form,

$$f_P(\beta) = \frac{C}{\sqrt{C_O C_P}} \int_{\mathbb{R}^2} e^{-iB(\beta, \lambda)} f_O(\lambda) d^2 \lambda. \quad (24)$$

Since both coefficient representations arise from Parseval-type expansions of the same Hilbert–Schmidt operator, the corresponding transform must be unitary on $L^2(\mathbb{R}^2)$, the forward and inverse coefficient maps must compose to the identity. Making use of the nondegeneracy of the bilinear form, we find

$$\begin{aligned} f_O(\lambda) &= \left(\frac{C}{\sqrt{C_O C_P}} \right)^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{iB(\beta, \lambda)} e^{-iB(\beta, \lambda')} f_O(\lambda') d^2\beta d^2\lambda' \\ &= \left(\frac{C}{\sqrt{C_O C_P}} \right)^2 \int_{\mathbb{R}^2} f_O(\lambda') \left[\int_{\mathbb{R}^2} e^{iB(\beta, \lambda - \lambda')} d^2\beta \right] d^2\lambda'. \end{aligned} \quad (25)$$

Writing $B(\beta, \lambda) = \beta^T M \lambda$ with $\det M \neq 0$, one has the Fourier completeness relation

$$\int_{\mathbb{R}^2} e^{iB(\beta, \lambda - \lambda')} d^2\beta = \frac{(2\pi)^2}{|\det M|} \delta^{(2)}(\lambda - \lambda'). \quad (26)$$

Therefore,

$$f_O(\lambda) = \frac{(2\pi)^2}{|\det M|} \left(\frac{C}{\sqrt{C_O C_P}} \right)^2 f_O(\lambda), \quad (27)$$

and consistency requires

$$\frac{C}{\sqrt{C_O C_P}} = \frac{\sqrt{|\det M|}}{2\pi}. \quad (28)$$

The transform between mutually unbiased coefficient amplitudes therefore becomes

$$f_P(\beta) = \frac{\sqrt{|\det M|}}{2\pi} \int_{\mathbb{R}^2} e^{-iB(\beta, \lambda)} f_O(\lambda) d^2\lambda, \quad (29)$$

with inverse

$$f_O(\lambda) = \frac{\sqrt{|\det M|}}{2\pi} \int_{\mathbb{R}^2} e^{iB(\beta, \lambda)} f_P(\beta) d^2\beta. \quad (30)$$

An equivalent condition follows by evaluating the relevant composition acting on $a_P(\beta)$. Therefore, once the overlap between the two operator bases is governed by the bilinear and nondegenerate phase $e^{-iB(\beta, \lambda)}$, the two coefficient representations of the same operator are naturally related as Fourier-dual functions with respect to the bilinear pairing B . In this sense, operator mutual unbiasedness induces a Fourier relation at the level of Hilbert–Schmidt coefficient functions.

In deriving an entropic relation between these two quantities, we first return to equation (22) and introduce the variable, $\eta \in \mathbb{R}^2$, given as,

$$\eta = M^T \beta \Rightarrow d^2\eta = |\det M| d^2\beta. \quad (31)$$

We now introduce the function, $\hat{f}_O(\eta)$, related to $f_O(\lambda)$ via the standard Fourier

$$\hat{f}_O(\eta) := \frac{1}{2\pi} \int_{\mathbb{R}^2} f_O(\lambda) e^{-i\eta^T \lambda} d^2\lambda, \quad (32)$$

with

$$\int_{\mathbb{R}^2} |\hat{f}_O(\eta)|^2 d^2\eta = 1. \quad (33)$$

We then define

$$H_{\hat{O}} = - \int_{\mathbb{R}^2} |\hat{f}_O(\eta)|^2 \ln |\hat{f}_O(\eta)|^2 d^2\eta, \quad (34)$$

and using the Hirschman–Beckner relation [3–5] for the Fourier normalization convention of equation (32) one immediately arrive at,

$$H_O + H_{\hat{O}} \geq 2 \ln(\pi e). \quad (35)$$

Thanks to equation (32) we can rewrite the Fourier relation of equation (29) as

$$f_P(\beta) = \sqrt{|\det M|} \widehat{f_O}(\eta), \quad (36)$$

and substituting this into H_P as defined in equation (10), we obtain

$$\begin{aligned} H_P &= - \int_{\mathbb{R}^2} |\widehat{f_O}(\eta)|^2 \ln |\widehat{f_O}(\eta)|^2 d^2\eta - \ln |\det M| \\ &= H_{\widehat{O}} - \ln |\det M|. \end{aligned} \quad (37)$$

Together with equation (35), we obtain our desired entropic relation,

$$H_O + H_P \geq 2 \ln(\pi e) - \ln |\det M|. \quad (38)$$

The inequality (38) constitutes a refinement of the general operator-frame entropic uncertainty relation obtained in section 2. Indeed, for mutually unbiased operator frames, $\sup_{\beta, \lambda} |K(\beta, \lambda)| = \sqrt{|\det M|}/2\pi$, so that the interpolation argument of section 2 yields a lower bound which is exceeded by (38) by the constant amount $2 \ln(e/2)$. This improvement arises from the additional Fourier structure induced by mutual unbiasedness, which permits the use of the sharp Hirschman–Beckner theorem rather than interpolation arguments alone.

It is worth noting that the entropic relation of (38) bears a formal resemblance to continuous-variable entropic uncertainty relations derived for families of observables [7, 8], where the entropies quantify the uncertainty associated with measurement outcomes and the lower bound is expressed in terms of the determinant of a commutator matrix. In the present framework, however, the entropies characterize the localization of operator representations in mutually unbiased operator frames rather than the statistics of measurement outcomes. Further to that, the determinant reflects the nondegeneracy of the bilinear overlap kernel defining mutual unbiasedness between operator families, which need not correspond to Hermitian observables. Interestingly, the dimensional contribution $2 \ln(\pi e)$ appearing in the phase-space realization coincides formally with the corresponding term arising in the two-dimensional observable setting of continuous-variable entropic uncertainty relations [7, 8]. In fact, the two relations concern fundamentally different quantities: in the present work, the entropy is defined over operator-frame coefficients on a two-dimensional label space, whereas in the observable setting it is defined over measurement probability distributions.

The inequality becomes particularly transparent in the canonical symplectic case $|\det M| = 1$, for which the lower bound reduces to $2 \ln(\pi e)$, identical to the standard two-dimensional Hirschman–Beckner bound.

4. Displacement operators and wigner kernels as a canonical realization

We now specialize the general operator-frame construction to the continuous-variable phase space $(\mathbb{R}^2, \Omega)$ equipped with its canonical symplectic structure, where Ω may be represented, in canonical coordinates, by a 2×2 antisymmetric matrix. In this setting, the displacement operators provide a natural continuous orthogonal operator frame associated with phase-space translations. As we shall show, the corresponding displaced parity operators form a mutually unbiased operator family whose overlap kernel is generated by the symplectic bilinear form. This realizes the general Fourier–bilinear framework developed earlier in a canonical phase-space setting and leads naturally to a symplectic Fourier duality between the two operator representations.

Let $D(\alpha)$ be the displacement operators [19],

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}, \quad \alpha \in \mathbb{C} \simeq \mathbb{R}^2 \quad (39)$$

and the family of operators $\{D(\alpha)\}$ be a continuous frame for $B_2(\mathcal{H})$ fulfilling the orthogonality relation

$$\text{Tr}[D(\alpha)D^\dagger(\alpha')] = \pi \delta^{(2)}(\alpha' - \alpha). \quad (40)$$

Let us further consider the case for an orthogonal basis of a family of displaced parity operators, namely the Wigner kernel, $\Delta(\beta)$, $\beta \in \mathbb{C} \simeq \mathbb{R}^2$. The Wigner phase-space representation admits a natural operator-valued kernel formulation; the Wigner function, $W(\beta)$, of a quantum state ρ is expressed as the expectation value of a displaced parity operator, $W(\beta) = \text{Tr}[\rho \Delta(\beta)]$, [20, 21]. The Wigner kernel is Hermitian

and unitary, with eigenvalues ± 1 , and corresponds physically to a local parity measurement about β and is defined as

$$\Delta(\beta) := 2D(\beta)\Pi D^\dagger(\beta), \quad (41)$$

where Π is the parity operator. The family $\{\Delta(\beta)\}$ forms an orthogonal, continuous operator frame and any operator can therefore be written in terms of the $\{\Delta(\beta)\}$ [20].

Let us consider the trace overlap

$$\text{Tr}[\Delta^\dagger(\beta)D(\alpha)] = 2e^\kappa \text{Tr}[D(\alpha)\Pi] \quad (42)$$

where $\kappa = \alpha\beta^* - \alpha^*\beta = 2i \text{Im}(\alpha^*\beta)$. Then,

$$\text{Tr}[D(\alpha)\Pi] = \int_{\mathbb{R}^2} \frac{d^2\zeta}{\pi} \langle \zeta | D(\alpha)\Pi | \zeta \rangle = \int_{\mathbb{R}^2} \frac{d^2\zeta}{\pi} \langle \zeta | D(\alpha) | -\zeta \rangle. \quad (43)$$

$$= \int_{\mathbb{R}^2} \frac{d^2\zeta}{\pi} \exp\left(-\frac{|\alpha|^2}{2}\right) \exp(-2|\zeta|^2 + \alpha\zeta^* + \alpha^*\zeta) = \frac{1}{2}. \quad (44)$$

Hence from equation (42),

$$|\text{Tr}[\Delta^\dagger(\beta)D(\alpha)]| = |2e^\kappa \text{Tr}[D(\alpha)\Pi]| = |e^\kappa| = 1. \quad (45)$$

Thus, the families $\{D(\alpha)\}$ and $\{\Delta(\beta)\}$ form a pair of mutually unbiased continuous operator frames as per our definition of equation (11) with the phase kernel being not only bilinear but symplectic. In the present case, the bilinear phase appearing in the overlap kernel is the symplectic form on $\mathbb{R}^2$. Considering $\alpha, \beta \in \mathbb{R}^2$, one has

$$\kappa = -2i\alpha^T\Omega\beta, \quad (46)$$

where $\det\Omega = 1$. Equivalently, the bilinear kernel B in this case may be written in the form $B(\alpha, \beta) = \alpha^T M \beta$ with $M = -2\Omega$, so that $|\det M| = 4$. The overlap kernel is therefore generated by a nondegenerate antisymmetric bilinear form associated with the canonical symplectic structure underlying continuous-variable phase space.

4.1. Entropic uncertainty between displacement and wigner kernel

Let $A_{D\Delta} \in B_2(\mathcal{H})$ be a Hilbert-Schmidt operator, such that

$$A_{D\Delta} = \frac{1}{\pi} \int_{\mathbb{R}^2} a_D(\alpha) D(\alpha) d^2\alpha = \frac{1}{\pi} \int_{\mathbb{R}^2} a_\Delta(\beta) \Delta(\beta) d^2\beta. \quad (47)$$

with the coefficient functions, $a_D(\alpha)$ and $a_\Delta(\beta)$

$$a_D(\alpha) := \text{Tr}[D^\dagger(\alpha)A_{D\Delta}], \quad a_\Delta(\beta) := \text{Tr}[\Delta^\dagger(\beta)A_{D\Delta}]. \quad (48)$$

The coefficient intensity $|a_D(\alpha)|^2$ defines a measure on $\mathbb{R}^2$ whose total mass is $\pi \|A\|_{\text{HS}}^2$. Let us define the normalized coefficient densities

$$f_D(\alpha) := \frac{a_D(\alpha)}{\sqrt{\pi} \|A_{D\Delta}\|_{\text{HS}}}, \quad f_\Delta(\beta) := \frac{a_\Delta(\beta)}{\sqrt{\pi} \|A_{D\Delta}\|_{\text{HS}}}, \quad (49)$$

which satisfy

$$\int_{\mathbb{R}^2} |f_D(\alpha)|^2 d^2\alpha = \int_{\mathbb{R}^2} |f_\Delta(\beta)|^2 d^2\beta = 1. \quad (50)$$

The corresponding Shannon differential entropies

$$H_D := - \int_{\mathbb{C}} |f_D(\alpha)|^2 \log |f_D(\alpha)|^2 d^2\alpha, \quad H_\Delta := - \int_{\mathbb{C}} |f_\Delta(\beta)|^2 \log |f_\Delta(\beta)|^2 d^2\beta,$$

together with $\det M = 4$, applying equation (38), give

$$H_D + H_\Delta \geq 2 \ln\left(\frac{\pi e}{2}\right).$$

The above uncertainty relation refers to the spreads of the coefficient distributions of $A_{D\Delta}$ in the corresponding representations. The inequality therefore implies that an operator cannot be simultaneously sharply concentrated in both the displacement and Wigner-kernel representations, reflecting the Fourier duality between these operator frames.

5. Position and momentum operators as mutually unbiased frames

The general operator-frame construction may also be realized through the operator families generated by the position and momentum eigenstates.

Let $\{|x\rangle\}_{x \in \mathbb{R}}$ and $\{|p\rangle\}_{p \in \mathbb{R}}$ denote the generalized eigenbases of the canonical position and momentum operators, understood in the usual rigged-Hilbert-space sense, satisfying

$$\langle x|x'\rangle = \delta(x-x'), \quad \langle p|p'\rangle = \delta(p-p'), \quad (51)$$

We now consider the operator families, $\{X(x, x')\}$ and $\{Y(p, p')\}$,

$$X(x, x') := |x\rangle\langle x'|, \quad Y(p, p') := |p\rangle\langle p'|, \quad (52)$$

which form generalized continuous operator frames for the Hilbert–Schmidt space $B_2(\mathcal{H})$. Their orthogonality relations are

$$\text{Tr}[X^\dagger(x, x')X(y, y')] = \delta(x-y)\delta(x'-y'), \quad (53)$$

$$\text{Tr}[Y^\dagger(p, p')Y(q, q')] = \delta(p-q)\delta(p'-q'). \quad (54)$$

Let us now evaluate the overlap between the two operator families.

$$\text{Tr}[Y^\dagger(p, p')X(x, x')] = \text{Tr}[|p'\rangle\langle p|x\rangle\langle x'|] \quad (55)$$

$$= \langle p|x\rangle\langle x'|p'\rangle. \quad (56)$$

Using the Fourier kernel relation between position and momentum eigenstates,

$$\langle p|x\rangle = \frac{1}{\sqrt{2\pi}}e^{-ipx}, \quad \langle x'|p'\rangle = \frac{1}{\sqrt{2\pi}}e^{ip'x'}, \quad (57)$$

we obtain

$$\text{Tr}[Y^\dagger(p, p')X(x, x')] = \frac{1}{2\pi}e^{-i(px-p'x')}. \quad (58)$$

Hence,

$$|\text{Tr}[Y^\dagger(p, p')X(x, x')]| = \frac{1}{2\pi}, \quad (59)$$

independent of (x, x') and (p, p') . The two operator families therefore form a pair of mutually unbiased continuous operator frames in the sense of equation (5). The overlap kernel is generated by the bilinear form, B , given as

$$B[(p, p'), (x, x')] = px - p'x', \quad (60)$$

which may be written in matrix form as

$$B(\beta, \lambda) = (p, p')M(x, x')^T, \quad \text{with } M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (61)$$

being a symmetric matrix of determinant -1 .

5.1. Position-momentum uncertainty principle

For a Hilbert–Schmidt operator $A_{XY} \in B_2(\mathcal{H})$, we may write

$$A_{XY} = \int_{\mathbb{R}^2} a_X(x, x') |x\rangle\langle x'| dx dx' = \int_{\mathbb{R}^2} a_Y(p, p') |p\rangle\langle p'| dp dp', \quad (62)$$

where

$$a_X(x, x') = \text{Tr}[X^\dagger(x, x') A_{XY}], \quad a_Y(p, p') = \text{Tr}[P^\dagger(p, p') A_{XY}]. \quad (63)$$

Introducing the normalized coefficient amplitudes

$$f_X(x, x') = \frac{a_X(x, x')}{\|A_{XY}\|_{\text{HS}}}, \quad f_Y(p, p') = \frac{a_Y(p, p')}{\|A_{XY}\|_{\text{HS}}} \quad (64)$$

one has

$$\int_{\mathbb{R}^2} |f_X(x, x')|^2 dx dx' = \int_{\mathbb{R}^2} |f_Y(p, p')|^2 dp dp' = 1. \quad (65)$$

The associated differential entropies are

$$\begin{aligned} H_X &= - \int_{\mathbb{R}^2} |f_X(x, x')|^2 \ln |f_X(x, x')|^2 dx dx', \\ H_Y &= - \int_{\mathbb{R}^2} |f_Y(p, p')|^2 \ln |f_Y(p, p')|^2 dp dp'. \end{aligned} \quad (66)$$

Making use of the general entropic relation of equation (38) with $|\det M| = 1$, we have

$$H_X + H_Y \geq 2 \ln(\pi e). \quad (67)$$

This inequality states that a Hilbert–Schmidt operator cannot be simultaneously sharply localized in both the position-dyadic representation and the momentum-dyadic representation.

5.2. Pure-state density operators and the connection to position–momentum uncertainty

It is immediately tempting to suspect a relationship between the entropic uncertainty relation derived above and the ordinary position–momentum uncertainty principle. To clarify the connection between the present operator–kernel formalism and the ordinary position–momentum uncertainty principle, consider a pure quantum state $\rho = |\psi\rangle\langle\psi|$. In the position representation, $\psi(x) = \langle x|\psi\rangle$, so that

$$|\psi\rangle = \int_{\mathbb{R}} \psi(x) |x\rangle dx, \quad \langle\psi| = \int_{\mathbb{R}} \psi^*(x') \langle x'| dx'. \quad (68)$$

The associated density operator therefore admits the expansion

$$\rho = \iint_{\mathbb{R}^2} \psi(x) \psi^*(x') |x\rangle\langle x'| dx dx'. \quad (69)$$

Hence, the coefficient kernel in the position-dyadic representation is given by

$$\rho(x, x') = \langle x|\rho|x'\rangle = \psi(x) \psi^*(x'). \quad (70)$$

Likewise, in the momentum representation,

$$\tilde{\rho}(p, p') = \langle p|\rho|p'\rangle = \tilde{\psi}(p) \tilde{\psi}^*(p'), \quad (71)$$

where

$$\tilde{\psi}(p) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \psi(x) e^{-ipx} dx. \quad (72)$$

Substituting the Fourier transform relation for $\tilde{\psi}(p)$ and its complex conjugate yields

$$\tilde{\rho}(p, p') = \frac{1}{2\pi} \iint_{\mathbb{R}^2} \rho(x, x') e^{-i(px - p'x')} dx dx'. \quad (73)$$

Thus, the dyadic Fourier relation appearing in the present framework is inherited directly from the ordinary Fourier relation between the position- and momentum-space wavefunction amplitudes. The diagonal components of the kernels recover the ordinary position and momentum probability densities,

$$\rho(x, x) = |\psi(x)|^2, \quad \tilde{\rho}(p, p) = |\tilde{\psi}(p)|^2. \quad (74)$$

Accordingly, the standard position–momentum entropic uncertainty relation ultimately originates from the Fourier duality between the position-space and momentum-space wavefunction amplitudes $\psi(x)$ and $\tilde{\psi}(p)$, with the associated probability densities $|\psi(x)|^2$ and $|\tilde{\psi}(p)|^2$ arising through the Born rule. In contrast, the present operator-level framework concerns the localization properties of the complete density-operator representations $\rho(x, x')$ and $\tilde{\rho}(p, p')$, including both diagonal probability contributions and off-diagonal coherence terms. The resulting entropic uncertainty relation therefore constrains not only the simultaneous localization of position and momentum probability distributions, but also the localization of the corresponding coherence structure encoded in the off-diagonal operator components.

6. Conclusion

The present work develops a general operator-frame framework for entropic uncertainty relations in Hilbert–Schmidt space. Beginning with continuous operator frames and their associated coefficient representations, we established a general entropic uncertainty relation by combining endpoint norm estimates with Riesz–Thorin interpolation. The entropic relation quantifies the extent to which two operator representations may be simultaneously localized.

We then identified a distinguished class of operator frames satisfying a mutual unbiasedness condition. Under natural assumptions of translational covariance and nondegeneracy, the overlap structure between such frames induces a bilinear Fourier transform relating their normalized coefficient amplitudes. The resulting construction unifies a variety of Fourier dualities encountered in continuous-variable quantum theory within a single framework, with different physical realizations arising from different choices of bilinear pairing. While arbitrary operator frames obey the general entropic uncertainty relation derived from interpolation theory, mutually unbiased operator frames possess a Fourier structure to satisfy a strictly stronger Hirschman–Beckner-type uncertainty relation. In this sense, mutually unbiased operator frames occupy a special position within the broader operator-frame framework, providing a natural operator-level analogue of Fourier duality and complementarity.

The displacement operators and Wigner kernels furnish a canonical continuous-variable realization of the construction. In this setting, the familiar relationship between characteristic-function and Wigner representations emerges as a concrete instance of the general theory, with the symplectic structure encoded through the bilinear pairing underlying the Fourier transform. Likewise, the Cartesian dyadic construction based on position and momentum eigenstates illustrates how the resulting operator-level framework naturally encompasses the ordinary position–momentum Fourier duality while extending beyond probability distributions to the full operator kernels, including their coherence structure.

A natural extension for future investigation is the exploration of additional realizations of mutually unbiased operator frames beyond the two-dimensional label spaces considered in the phase-space and Cartesian constructions presented here. More generally, one may consider operator frames indexed by higher-dimensional parameter spaces (corresponding for example, to multimode continuous-variable systems) or generated from alternative families of continuous-variable eigenstates. Such extensions may reveal new forms of operator-level complementarity and associated entropic uncertainty relations. An interesting future direction would also include the construction of operator frames from rotated quadrature eigenstates [7, 8]. Since distinct quadrature eigenbases possess constant-modulus overlaps, the associated dyadic operator families may provide natural candidates for further realizations of the mutual unbiasedness structure considered here. In such settings, the diagonal sectors of the corresponding operator kernels yield the probability distributions associated with quadrature measurements. It would therefore be of interest to investigate whether observable-level entropic uncertainty relations, such as those obtained for rotated quadratures [7, 8], can be understood as arising from diagonal restrictions of a more general operator-frame framework. Such a connection would further illuminate the relationship between conventional uncertainty relations for measurement outcomes and the broader operator-level complementarity developed in the present work.

Data availability statement

The data cannot be made publicly available upon publication because no suitable repository exists for hosting data in this field of study. The data that support the findings of this study are available upon reasonable request from the authors.

Appendix

The generalized orthogonality and reconstruction relations in equations (1) and (3), together with the trace pairings in equation (4), define the coefficient maps

$$A \mapsto a_O(\lambda) = \text{Tr}[O^\dagger(\lambda)A], \quad A \mapsto a_P(\beta) = \text{Tr}[P^\dagger(\beta)A], \quad (75)$$

which are rescaled isometries from $B_2(\mathcal{H})$ into $L^2(\mathbb{R}^2)$ via the corresponding Parseval relations,

$$\|A\|_{\text{HS}}^2 = \frac{1}{C_O} \int_{\mathbb{R}^2} |a_O(\lambda)|^2 d^2\lambda = \frac{1}{C_P} \int_{\mathbb{R}^2} |a_P(\beta)|^2 d^2\beta. \quad (76)$$

This also ensures the normalization of f_O and f_P as defined in equation (5). We restrict attention to coefficient functions f_O for which the differential Shannon entropy exists and which admit an approximating sequence of test functions, $\{\phi_O^{(n)}\} \subset \mathcal{D}$, where $\mathcal{D}$ is a dense subspace contained in $L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ satisfying

$$\|\phi_O^{(n)} - f_O\|_2 \rightarrow 0. \quad (77)$$

The L^2 convergence ensures that the approximating sequence represents the same limiting coefficient function. Since T maps $\mathcal{D}$ into itself, we could define a corresponding sequence via, $T\phi_O^{(n)} = \phi_P^{(n)}$, as $\{\phi_P^{(n)}\}_{n=1}^\infty \subset \mathcal{D}$. In order to apply the Riesz–Thorin interpolation theorem [18], we consider the representative $\phi_O^{(n)} \in \mathcal{D}$ (and the corresponding $\phi_P^{(n)}$), and we begin with estimating an endpoint via equation (7),

$$\begin{aligned} |\phi_P^{(n)}(\beta)| &= \left| \int K(\beta, \lambda) |\phi_O^{(n)}(\lambda)| d^2\lambda \right| \leq \int |K(\beta, \lambda)| |\phi_O^{(n)}(\lambda)| d^2\lambda \\ &\leq C_K \int |\phi_O^{(n)}(\lambda)| d^2\lambda = C_K \|\phi_O^{(n)}\|_1, \end{aligned} \quad (78)$$

with $C_K = \sup_{\beta, \lambda} |K(\beta, \lambda)|$. Taking the supremum over β gives $\|T\phi_O^{(n)}\|_\infty \leq C_K \|\phi_O^{(n)}\|_1$. This establishes one endpoint bound. Equation (6) which gives $\|\phi_O^{(n)}\|_2 = \|\phi_P^{(n)}\|_2$, establishes another endpoint bound, $\|T\phi_O^{(n)}\|_2 = \|\phi_O^{(n)}\|_2$. Applying the Riesz–Thorin interpolation theorem [18] to the endpoint bounds yields

$$\|\phi_P^{(n)}\|_q \leq C_K^{1-\theta} \|\phi_O^{(n)}\|_p, \quad (79)$$

where

$$\frac{1}{p} = 1 - \frac{\theta}{2}, \quad \frac{1}{q} = \frac{\theta}{2}, \quad 0 \leq \theta \leq 1. \quad (80)$$

Eliminating θ , these relations give $q = p/(p-1)$ and $1 - \theta = 2/p - 1$. Hence

$$\|\phi_P^{(n)}\|_q \leq C_K^{\frac{2}{p}-1} \|\phi_O^{(n)}\|_p, \quad 1 \leq p \leq 2, \quad \frac{1}{p} + \frac{1}{q} = 1. \quad (81)$$

Taking logarithms of equation (81) gives

$$\ln \|\phi_P^{(n)}\|_q \leq \left(\frac{2}{p} - 1 \right) \ln C_K + \ln \|\phi_O^{(n)}\|_p, \quad (82)$$

with $q = p/(p-1)$. Differentiating equation (82) with respect to p and evaluating at $p = q = 2$ yields

$$H(|\phi_O^{(n)}|^2) + H(|\phi_P^{(n)}|^2) \geq -2 \ln C_K, \quad (83)$$

where H denotes the Shannon differential entropy. We note that we have assumed the finiteness of the Shannon entropy for the test functions. The regular coefficient functions considered here include a broad class of smooth and well-behaved distributions that arise in physically relevant representations. To extend the result to a general coefficient function $f_O \in L^2(\mathbb{R}^2)$, however, L^2 -convergence alone is insufficient to guarantee convergence of the differential Shannon entropy. Such convergence is ensured under additional regularity or integrability conditions on the coefficient densities and their approximating sequences; several sufficient conditions are known in the literature [22–25]. As a particular example, Piera and Parada [23] provide sufficient conditions based on convergence in variation together with appropriate boundedness conditions on the probability densities, providing one possible route to the required entropy convergence. Accordingly, the extension to f_O is understood for coefficient functions and approximating sequences for which

$$H(|\phi_O^{(n)}|^2) \longrightarrow H(|f_O|^2), \quad (84)$$

and analogously for the P -representation.

ORCID iDs

Jesni Shamsul Shaari  0000-0002-3733-7550

Stefano Mancini  0000-0002-3797-3987

References

- [1] Heisenberg W 1927 Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik *Z. Phys* vol **43** 172–98
Heisenberg W 1983 The physical content of quantum kinematics and mechanics *Quantum Theory and Measurement*, ed J A Wheeler and W H Zurek (Princeton University Press) pp 62–84
- [2] Robertson H P 1929 The uncertainty principle *Phys. Rev.* **34** 163–4
- [3] Hirschman I I 1957 A note on entropy *Am. J. Math.* **79** 152
- [4] Beckner W 1975 Inequalities in Fourier analysis *Ann. Math.* **102** 159
- [5] Białynicki-Birula I and Mycielski J 1975 Uncertainty relations for information entropy in wave mechanics *Commun. Math. Phys.* **44** 129
- [6] Coles P J, Berta M, Tomamichel M and Wehner S 2017 Entropic uncertainty relations and their applications *Rev. Mod. Phys.* **89** 015002
- [7] Hertz A, Vanbever L and Cerf N J 2018 Multidimensional entropic uncertainty relation based on a commutator matrix in position and momentum spaces *Phys. Rev. A* **97** 012111
- [8] Hertz A and Cerf N J 2019 Continuous-variable entropic uncertainty relations *J. Phys. A: Math. Theor.* **52** 173001
- [9] Ricaud B and Torrèsani B 2013 Refined support and entropic uncertainty inequalities *IEEE Trans. Inf. Theory* **59** 4272–9
- [10] Maassen H and Uffink J B M 1988 Generalized entropic uncertainty relations *Phys. Rev. Lett.* **60** 1103–6
- [11] Schwinger J 1960 Unitary operator bases *Proc. Natl Acad. Sci. USA* **46** 570–9
- [12] Durt T, Bengtsson I, Englert B-G and Życzkowski K 2010 On mutually unbiased bases *Int. J. Quantum Inf.* **8** 535–640
- [13] Weigert S and Wilkinson M 2008 Mutually unbiased bases for continuous variables *Phys. Rev. A* **78** 020303(R)
- [14] Scott A J 2008 Optimizing quantum process tomography with unitary 2-designs *J. Phys. A* **41** 055308
- [15] Shaari J S, Nasir R N M and Mancini S 2016 Mutually unbiased unitary bases *Phys. Rev. A* **94** 052328
- [16] Shaari J S and Mancini S 2020 Entropic bounds for unitary testers and mutually unbiased unitary bases *Ann. Phys., NY* **412** 168043
- [17] Shaari J S, Nasir R N M and Mancini S 2021 Entropic bounds as uncertainty measure of unitary operators *Phys. Lett. A* **398** 127277
- [18] Reed M and Simon B 1975 *Methods of Modern Mathematical Physics II (Fourier Analysis, Self-Adjointness)* (Academic)
- [19] Cahill K E and Glauber R J 1969 Ordered expansions in boson amplitude operators *Phys. Rev.* **177** 1857
- [20] Cahill K E and Glauber R J 1969 Density operators and quasiprobability distributions *Phys. Rev.* **177** 1882
- [21] Royer A 1977 Wigner function as the expectation value of a parity operator *Phys. Rev. A* **15** 449
- [22] Godavarti M and Hero A O 2004 Convergence of differential entropies *IEEE Trans. Inf. Theory* **50** 171
- [23] Piera F J and Parada P 2009 On convergence properties of Shannon entropy *Probl. Inf. Transm.* **45** 75–94
- [24] Ghourchian H, Gohari A and Amini A 2017 Existence and continuity of differential entropy for a class of distributions *IEEE Commun. Lett.* **21** 1469
- [25] Godavarti M 2026 Convergence of differential entropies – II (arXiv: 2602.13493)