RESEARCH ARTICLE | JUNE 09 2026

Consensus convergence of multi-agent systems controlled via geometric quadratic stochastic operators

Siti Nurlaili Karim; Nur Syahirah Roslan; Sarah A'fifah Abdullah Sani; Nur Syadhila Che Lah; Nur Zatul Akmar Hamzah

AIP Conf. Proc. 3444, 070001 (2026)
https://doi.org/10.1063/5.0327164

Consensus Convergence of Multi-Agent Systems Controlled via Geometric Quadratic Stochastic Operators

Siti Nurlaili Karim,[1, a)] Nur Syahirah Roslan,[1, b)] Sarah A'fifah Abdullah Sani,[2, c)] Nur Syadhila Che Lah,[3, d)] and Nur Zatul Akmar Hamzah[4, e)]

[1)] *Department of Computer Science, Faculty of Information and Communication Technology, Universiti Tunku Abdul Rahman, Jalan Universiti, Bandar Barat, 31900 Kampar, Perak, Malaysia.*
[2)] *Faculty of Computing and Informatics, Multimedia University, Persiaran Multimedia, 63100 Cyberjaya, Selangor, Malaysia.*
[3)] *Universiti Teknologi Petronas, 32610 Seri Iskandar, Perak, Malaysia.*
[4)] *Computational and Theoretical Sciences Department, Kulliyyah of Science, International Islamic University Malaysia, Jalan Sultan Ahmad Shah, Bandar Indera Mahkota, 25200 Kuantan, Pahang, Malaysia.*

[a)] *Corresponding author: nurlaili.karim@utp.edu.my*
[b)] *syahirahr@utar.edu.my*
[c)] *sarah.afifah@mmu.edu.my*
[d)] *nursyadhila.chelah@utp.edu.my*
[e)] *zatulakmar@iium.edu.my*

Abstract. Consensus control in multi-agent systems has emerged as a key domain of research, owing to its broad implementations in numerous fields. However, achieving consensus, where all agents agree on a common value, presents a key challenge in the study of multi-agent coordination. To address this, consensus models are formulated and developed specifically to solve such problems. Previous empirical and theoretical studies have shown that nonlinear models outperform linear models in tackling consensus challenges. Limited studies of the nonlinear consensus models promote the idea of a potential nonlinear model utilising Geometric quadratic stochastic operators (QSOs). This paper proposed Geometric QSOs for solving consensus problems in the systems of multi-agent. For consensus convergence, the suggested method makes advantage of a nonlinear class of family of QSOs with geometric distribution. The foundation of the nonlinear protocols for Geometric QSOs is probability and measure theory. The study uses geometric QSOs to examine how the multi-agent systems converge to the ideal value. An investigation was conducted in comparison between the DeGroot model and current nonlinear models. Our results suggest that the suggested nonlinear consensus model can be used to solve a more sophisticated consensus problem.

INTRODUCTION

In recent years, extensive research on multi-agent systems (MAS) has concentrated on addressing the challenges faced by autonomous groups of agents in distributed systems. A major issue is getting all agents to agree on a common value, known as consensus. Achieving consensus means all agents must reach the same status, which is a significant challenge in coordinating multi-agents. To achieve this, different connectivity requirements and alternating topologies are used [1]. The consensus problem involves getting multiple independent agents to agree through local interactions, meaning all agents must have equal statuses.

The consensus problem is important in many fields, like robotics, distributed computing, and sensor networks [2, 3, 4]. In multi-agent systems, creating a framework that ensures all agents reach a consensus is a big challenge. Agents' states can include views, parameters, estimates, doctrines, locations, and velocities.

DeGroot has significantly contributed to solving the consensus problem [5]. His model shows that initial value vectors, representing viewpoints, are crucial for achieving consensus [6]. A graphical method for achieving consensus in dynamic environments using a linear model has been explored [7]. Furthermore, the consensus problem in linear time with a stochastic matrix containing positive diagonal entries has been investigated [8]. Another study looked at the dynamics of a linear general consensus with arbitrary distribution, governed by pattern-triggered events [9]. However, these studies assume that agents are linear, which is often not the case in real-world physical systems [10].

Many studies have tried to include nonlinear convergence protocols in the MAS consensus problem. Designing a nonlinear model for many network nodes is a complex mathematical task. Various theoretical and empirical works have explored this, including the use of doubly stochastic matrices [11]. Early work on nonlinear stability control has also been discussed. Both linear and nonlinear approaches have been proposed for achieving consensus in distributed systems [12]. Nonlinear consensus procedures are used in applications like active agents, which have entry constraints [13]. Most published nonlinear models are very complex and have limited constraints, such as cubic stochastic systems where each agent is stochastically related to every other agent. Current research focuses on using majorization theory

Proceedings of the International Conference on Mathematical Sciences and Technology 2024 (MathTech2024)
AIP Conf. Proc. 3444, 070001-1–070001-8; https://doi.org/10.1063/5.0327164
Published under an exclusive license by AIP Publishing. 978-0-7354-5378-4/$30.00

to understand the computational complexity of nonlinear consensus [14, 15]. It is beneficial to explore new nonlinear consensus models using measure and probability theory, which offer more possible contexts and groups of agents.

Consensus models fall into two types: linear and nonlinear models, according to earlier studies. While the nonlinear models are considerably more challenging to formulate but arrive at consensus more quickly, the linear model is comparatively simpler to construct but takes longer for reaching consensus. Considered to be the first linear consensus model, the DeGroot model was first proposed in 1974 as the most fundamental mathematical computational design for the consensus process. In the meanwhile, fractional orders, nonlinear stochastic procedures, the Lyapunov function, and nonlinear models with first and second orders are all included in the nonlinear consensus model. The nonlinear models outperform the linear model, as demonstrated in [16]. However, nonlinear models have the drawback of often being more complex and having limited configurations. Investigating nonlinear models with quicker convergence to an optimal complexity consensus that is still comparatively low and more durable is the main focus at the moment.

The nonlinear differential equations known as quadratic stochastic operators (QSOs), initially laid out by Bernstein [17], developed from particular population genetics issues. The main outcome of the enhanced class of QSO on a 2-dimensional simplex was established by Vallander [18]. Later, Lyubich [19] offered a thorough analysis of QSOs defined on a one-dimensional simplex. In [20], the doubly stochastic quadratic operators (DSQOs) notion was explained, along with the definition of the appropriate and adequate circumstances for these operators. Even on a two-dimensional simplex, the use of QSOs, which are first explored in population genetics, remains an open, challenging subject.

Some nonlinear consensus protocols have been created utilizing the QSO model [14, 15]. These protocols are often created to reach consensus quickly and with minimal complexity. Consensus issues in the systems of multi-agents can be effectively addressed by a number of methods. According to this viewpoint, we will address distributed system consensus issues by employing a subclass of QSOs called Geometric QSOs, which integrate measure and probability theory and have been the subject of recent, active research [21, 22, 23].

This paper is structured as follows. We presented our methodology through preliminaries of Geometric QSOs in Section 2. The DeGroot model, QSOs model, DSQOs model, and Geometric QSOs model involved for MAS are all covered in Section 3. We discussed the simulated comparison of the aforementioned models in Section 4. Section 5 serves as the paper's final conclusion.

METHODOLOGY

This section outlines the fundamental concepts, definitions, and formulations related to QSOs and introduces the Geometric QSO. The mathematical framework is built upon probability and measure theory, focusing on the dynamics of multi-agent systems.

We denote $(X,\mathscr{F})$ as a measurable space with a state space X and a σ-algebra, $\mathscr{F}$. Then, on the measurable space, we describe a set of all probability measures $S(X,\mathscr{F})$. For a finite state space $X=\{1,2,\ldots,m\}$, the corresponding σ-algebra is represented by a power set $P(X)$. The collection of all probability measures on $(X,P(X))$ forms an $(m-1)$-dimensional simplex, expressed as follows:

$$S^{m-1}=\left\{\mathbf{x}=(x_1,...,x_m)\in\mathbb{R}^m : x_i\geq 0,\ \sum_{i=1}^{m}x_i=1\right\}.$$

for $i=1,\ldots,m$.

Definition 1 . *A QSO is a transformation $V: S^{m-1}\to S^{m-1}$ defined as:*

$$(Vx)_k=\sum_{i,j=1}^{m}P_{ij,k}x_ix_j, \tag{1}$$

where the coefficients $P_{ij,k}$ represents a probability measure, satisfying the following conditions:

$$P_{ij,k}\geq 0,\ P_{ij,k}=P_{ji,k},\ \sum_{k=1}^{m}P_{ij,k}=1, \tag{2}$$

for all $i,j,k\in X$.

Given an arbitrary $\mathbf{x}^{(0)} \in S^{m-1}$, the trajectory of $\mathbf{x}^{(n)}$ generated by $\mathbf{x}^{(0)}$ under the operator in Eq. (1) is expressed as $\mathbf{x}^{(n+1)} = V(\mathbf{x}^{(n)})$, for $n = 0, 1, \ldots$. Alternatively, for any origin point indicated by $\lambda \in S(X, \mathscr{F})$, where $V^{n+1}\lambda = V(V^n\lambda)$, the sequence $\{V^n\lambda : n = 0, 1, ...\}$ represents the trajectory of λ.

Definition 2 . *If the limit*

$$\lim_{n\to\infty} V^n(\lambda)$$

exists for any initial point $\lambda \in S(X, \mathscr{F})$, then a QSO V is regular.

Throughout this study, we consider the Geometric QSO as the nonlinear consensus protocol. For the Geometric QSO, the following notions are necessary to be mentioned.

Consider a countable sample space $X = \{0, 1, \ldots\}$ with its corresponding σ-algebra $\mathscr{F}$, which is the power set $P(X)$. A probability measure μ is defined on the sample space X such that $\mu(\{k\}) = \mu(k)$, where $\mu(k)$ denotes the measure of each singleton $\{k\}$ with $k \in X$. For this countable state space X, the QSO V can be expressed in the following form:

$$V\mu(k) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} P_{ij,k}\mu(i)\mu(j), \tag{3}$$

where $\mu \in S(X, \mathscr{F})$ and $P_{ij,k}$ accommodates conditions in Eq. (2). A Geometric distribution G_r, defined on the countable set X, is characterized by a real parameter r, where $0 < r < 1$, and is given as:

$$G_r(k) = (1-r)r^k,\ k \in X.$$

Definition 3 . *A QSO V as defined in Eq. (3) is referred to as a Geometric QSO, if for any $i, j \in X$, the probability measure $P(i, j, \cdot)$ follows a geometric distribution with parameter $r(i, j)$, where $0 < r(i, j) < 1$.*

The Geometric QSO was first introduced in [21]. The behavior of Geometric QSOs generated by finite partitions was extensively investigated in [22, 23]. One can describe the Geometric QSO generated by m-partition as follows.

We consider a measurable m-partition of an infinitely countable set X, denoted as $\xi = \{A_1, A_2, \ldots, A_m\}$. Correspondingly, we define a partition on $X \times X$ as $\zeta = \{B_{ij} : i, j = 1, \ldots, m\}$, where $B_{ii} = A_i \times A_i$ for $i = 1, \ldots, m$, and $B_{ij} = (A_i \times A_j) \cup (A_j \times A_i)$ for $i \neq j$. Consequently, it holds that $B_{ij} = B_{ji}$. Next, we introduce a probability measures family μ_{ij} defined on the measurable space $(X, \mathscr{F})$, where $\mathscr{F}$ represents the power set of X. Using this setup, we define $P(x, y, A)$, indicating the probability measure as follows:

$$P(x,y,A) = \mu_{ij}(A) \text{ for } (x,y) \in B_{ij}. \tag{4}$$

For any $\lambda \in S(X, \mathscr{F})$ and measurable set $A \in \mathscr{F}$, the following holds:

$$(V\lambda)(A) = \int_X\int_X P(x,y,A)\,d\lambda(x)\,d\lambda(y) = \sum_{i,j=1}^{m}\int_{A_i}\int_{A_j}\mu_{ij}(A)\,d\lambda(x)\,d\lambda(y) = \sum_{i,j=1}^{m}\mu_{ij}(A)\lambda(A_i)\lambda(A_j).$$

Let λ represent an initial measure, and consider the trajectory $\{V^n\lambda : n = 0, 1, 2, \ldots\}$ generated by this measure. Here, the trajectory is defined recursively as $V^{n+1}\lambda = V(V^n\lambda)$ for all $n = 0, 1, 2, \ldots$, with the initial condition $V^0\lambda = \lambda$. Then,

$$\left(V^{n+1}\lambda\right)(A) = \sum_{i,j=1}^{m}\mu_{ij}(A)(V^n\lambda)(A_i)(V^n\lambda)(A_j),$$

with

$$\left(V^{n+1}\lambda\right)(A_k) = \sum_{i,j=1}^{m} \mu_{ij}(A_k)\left(V^n\lambda\right)(A_i)\left(V^n\lambda\right)(A_j), \tag{5}$$

for $k = 1,\ldots,m$.

Assume that $x_k{}^{(n)} = (V^n\lambda)(A_k)$ and $P_{ij,k} = \mu_{ij}(A_k)$. Then, for the vector $\left(x_1{}^{(n)},...,x_m{}^{(n)}\right) \in S^{m-1}$, we can express the system of equations in Eq. (5) in the form:

$$(Wx)_k = \sum_{i,j=1}^{m} P_{ij,k}x_ix_j, \tag{6}$$

for all $k = 1,\ldots,m$, satisfying conditions in Eq. (2).

According to the Geometric QSO analysis in [23], the 2-partition defined combines a finite-element and an infinite-element partition. It has been established that such operators can be either regular with a single fixed point or non-regular with period-2 periodic points. However, in this article, we will look into the Geometric QSO, which can converge to the center of a simplex. Consider the set $intS^{m-1} = \left\{\mathbf{x} \in S^{m-1} : \sum_{i=1}^{m} x_i = 1, x_i > 0\right\}$ as the simplex interior. The simplex vertices are denoted by points $e_k = \left\{0,0,\ldots,\frac{1}{k},\ldots,0\right\}$, while the scalar vector $\left(\frac{1}{m},\frac{1}{m},\ldots,\frac{1}{m}\right)$ is indicated as the simplex center.

In [21], it has been shown that the Geometric QSO with three different parameters, which is defined as follows:

$$P_{ij,k} = \begin{cases} (1-r_1)\,r_1^k & \text{if } i+j \equiv 0\,(\mathrm{mod}\,3), \\ (1-r_2)\,r_2^k & \text{if } i+j \equiv 1\,(\mathrm{mod}\,3), \\ (1-r_3)\,r_3^k & \text{if } i+j \equiv 2\,(\mathrm{mod}\,3), \end{cases} \tag{7}$$

is potentially converging to $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$, which is the center of the 2-dimensional simplex for some values of r_1, r_2, and r_3. Therefore, we shall consider such a definition of $P_{ij,k}$ for the intended nonlinear consensus protocol in this work.

RESULTS AND DISCUSSION

We present an analogous analysis of the DeGroot model, QSO model, DSQO model, and the proposed Geometric QSO model in this section to evaluate their performance in addressing consensus problems in MAS. For this analysis, three agents x, y, and z are considered, and their convergence behaviors are studied under different initial conditions.

DeGroot linear model

The DeGroot model is ruled by the iterative equation in the following form:

$$x^{(t+1)} = P(t)\,x(t) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where $P(t)$ represents a transition matrix, $x(t)$ is the vector of agent states at time t, and $(t+1)$ denotes the subsequent iteration. Here, $x(t+1)$ represents the updated state vector at iteration $(t+1)$, while the updated value of each individual components x, y, and z, calculated using the equation:

$$\begin{aligned} X &= a_{11}x + a_{12}y + a_{13}z, \\ Y &= a_{21}x + a_{22}y + a_{23}z, \\ Z &= a_{31}x + a_{32}y + a_{33}z, \end{aligned}$$

respectively, subject to the conditions:

$$a_{ij} \geq 0, \sum_{i=1}^{3} a_{1i} = \sum_{i=1}^{3} a_{2i} = \sum_{i=1}^{3} a_{3i} = 1, \sum_{i=1}^{3} a_{i1} = \sum_{i=1}^{3} a_{i2} = \sum_{i=1}^{3} a_{i3} = 1.$$

QSO nonlinear model

The QSO nonlinear model employs a more complex structure defined as:

$$(V\mathbf{x}) = \sum_{i,j=1}^{m} P_{ij,k}x_i x_j = x_i P_{ij,1}x_j + x_i P_{ij,2}x_j + \cdots + x_i P_{ij,m}x_j, \tag{8}$$

for $k = 1, \ldots, m$, where $P_{ij,k}$ represents the probability measures with x_i and x_j as agents vector column and row, respectively, in the form:

$$\sum_{i,j=1}^{m} P_{ij,k}x_i x_j = \begin{pmatrix} \begin{pmatrix} x & y & z \end{pmatrix} & \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} & \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ \begin{pmatrix} x & y & z \end{pmatrix} & \begin{pmatrix} a_{12} & a_{13} & a_{11} \\ a_{22} & a_{23} & a_{21} \\ a_{32} & a_{33} & a_{31} \end{pmatrix} & \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ \begin{pmatrix} x & y & z \end{pmatrix} & \begin{pmatrix} a_{13} & a_{11} & a_{12} \\ a_{23} & a_{21} & a_{22} \\ a_{33} & a_{31} & a_{32} \end{pmatrix} & \begin{pmatrix} x \\ y \\ z \end{pmatrix} \end{pmatrix},$$

for $i, j, k = 1, 2, 3$, where $x_1 = x$, $x_2 = y$, and $x_3 = z$. Then, the system of equations for the QSOs nonlinear operator can be presented as:

$$\begin{aligned} V_{(x)} &= a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + a_{21}xy + a_{31}xz + a_{12}xy + a_{32}yz + a_{13}xz + a_{23}yz, \\ V_{(y)} &= a_{12}x^2 + a_{23}y^2 + a_{31}z^2 + a_{22}xy + a_{32}xz + a_{13}xy + a_{33}yz + a_{11}xz + a_{21}yz, \\ V_{(z)} &= a_{13}x^2 + a_{21}y^2 + a_{32}z^2 + a_{23}xy + a_{33}xz + a_{11}xy + a_{31}yz + a_{12}xz + a_{22}yz. \end{aligned}$$

DSQO nonlinear model

A subclass of QSO, the DSQO model, simplifies the polynomial structure by incorporating majorization theory. This leads to reduced computational overhead as given in Eq. (8). Based on the model, the system of equations of the DSQOs nonlinear model can be represented by the following:

$$\sum_{i,j=1}^{m} P_{ij,k}x_i x_j = \begin{pmatrix} \begin{pmatrix} x & y & z \end{pmatrix} & \begin{pmatrix} a_{11,1} & a_{12,1} & a_{13,1} \\ a_{21,1} & a_{22,1} & a_{23,1} \\ a_{31,1} & a_{32,1} & a_{33,1} \end{pmatrix} & \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ \begin{pmatrix} x & y & z \end{pmatrix} & \begin{pmatrix} a_{12,2} & a_{13,2} & a_{11,2} \\ a_{22,2} & a_{23,2} & a_{21,2} \\ a_{32,2} & a_{33,2} & a_{31,2} \end{pmatrix} & \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ \begin{pmatrix} x & y & z \end{pmatrix} & \begin{pmatrix} a_{13,3} & a_{11,3} & a_{12,3} \\ a_{23,3} & a_{21,3} & a_{22,3} \\ a_{33,3} & a_{31,3} & a_{32,3} \end{pmatrix} & \begin{pmatrix} x \\ y \\ z \end{pmatrix} \end{pmatrix}.$$

The DSQO model's transition matrices:

$$P_{ij,1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, P_{ij,2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, P_{ij,3} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

satisfy specific doubly stochastic conditions by using majorization theory as in [24]. Hence, we will have the following evaluation of the nonlinear operator of DSQO:

$$\begin{aligned} V_{(x)} &= y^2 + 2xz, \\ V_{(y)} &= x^2 + 2yz, \\ V_{(z)} &= z^2 + 2xy. \end{aligned} \tag{9}$$

Geometric QSO nonlinear model

Lastly, we will introduce the protocols of the nonlinear Geometric QSO model, which represents a specific subclass of QSOs. Consider a Geometric QSO in the Definition 3 with the conditions of $P_{ij,k}$ in Eq. (7).

Let $A(\mu) = \sum_{i=0}^{\infty} \mu(3i)$, $B(\mu) = \sum_{i=0}^{\infty} \mu(3i+1)$, and $C(\mu) = \sum_{i=0}^{\infty} \mu(3i+2)$ for any initial measure $\mu \in S(X, \mathscr{F})$, where $A(\mu) + B(\mu) + C(\mu) = 1$. In this context, $A(\mu)$, $B(\mu)$, and $C(\mu)$ refer to the agent x , y, and z, respectively. One can easily verify that for parameter $0 < r < 1$, we have the following:

$$A(G_r) = \frac{1}{r^2 - 3r + 3}, B(G_r) = \frac{1-r}{r^2 - 3r + 3}, C(G_r) = \frac{(1-r)^2}{r^2 - 3r + 3}.$$

By implying the condition in Eq. (7) into the QSO in Eq. (3), the recurrent equation is represented by:

$$\begin{aligned} V^{n+1}\mu(k) = {} & r_1 (1-r_1)^k \left(A^2 (V^n\mu) + 2B(V^n\mu) C(V^n\mu)\right) + r_2 (1-r_2)^k \left(C^2 (V^n\mu) + 2A(V^n\mu) B(V^n\mu)\right) \\ & + r_3 (1-r_3)^k \left(B^2 (V^n\mu) + 2A(V^n\mu) C(V^n\mu)\right), \end{aligned} \tag{10}$$

with the following system of equations:

$$\begin{aligned} x' &= A(G_{r_1}) x^2 + A\left(G_{r_3}\right) y^2 + A(G_{r_2}) z^2 + 2A(G_{r_2}) xy + 2A\left(G_{r_3}\right) xz + 2A(G_{r_1}) yz, \\ y' &= B(G_{r_1}) x^2 + B\left(G_{r_3}\right) y^2 + B(G_{r_2}) z^2 + 2B(G_{r_2}) xy + 2B\left(G_{r_3}\right) xz + 2B(G_{r_1}) yz, \\ z' &= C(G_{r_1}) x^2 + C\left(G_{r_3}\right) y^2 + C(G_{r_2}) z^2 + 2C(G_{r_2}) xy + 2C\left(G_{r_3}\right) xz + 2C(G_{r_1}) yz, \end{aligned} \tag{11}$$

to represent $A(V^{n+1}\mu)$, $B(V^{n+1}\mu)$, and $C(V^{n+1}\mu)$, respectively with $x > 0$, $y > 0$, $z > 0$, and $x + y + z = 1$.

SIMULATION

To compare the four models discussed, we provide the following example. Consider the transition matrix used for both the DeGroot model and the QSO models as follows:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 0.2 & 0.6 & 0.2 \\ 0.5 & 0.15 & 0.35 \\ 0.3 & 0.25 & 0.45 \end{pmatrix} \tag{12}$$

We run three simulations with three different sets of initial values for the DeGroot model, QSOs model, DSQOs model, and Geometric QSO model as shown in Table 1, Table 2, and Table 3. We refer to the operator in Eq. (9) for the simulations of the DSQO model. Meanwhile, for the Geometric QSO model, one may apply the operator in Eq. (11) with parameters r_1, r_2, and r_3. Note that the values of r_1, r_2, and r_3 in these simulations are chosen based on their potential to converge closely to the simplex center.

TABLE 1. Iteration number to achieve convergence with $(x_0, y_0, z_0) = (0.5, 0.3, 0.2)$

	Fixed point of the convergence			
Consensus model	x	y	z	Number of iterations
DeGroot	0.333333	0.333333	0.333333	13
QSO	0.333333	0.333333	0.333333	3
DSQO	0.333333	0.333333	0.333333	4
Geometric QSO [a]	0.333934	0.333333	0.333273	2

[a] $r_1 = 0.9991$, $r_2 = 0.9973$, and $r_3 = 0.9982$.

One may notice that the Geometric QSO has an upper hand over all three other models, where it converges towards a fixed point with not more than three iterations for all three simulations. However, it was observed that the nonlinear

TABLE 2. Iteration number to achieve convergence with $(x_0, y_0, z_0) = (0.125, 0.75, 0.125)$

	Fixed point of the convergence			
Consensus model	x	y	z	Number of iterations
DeGroot	0.333333	0.333333	0.333333	15
QSO	0.333333	0.333333	0.333333	4
DSQO	0.333333	0.333333	0.333333	6
Geometric QSO [a]	0.33418	0.333332	0.332489	3

[a] $r_1 = 0.9989$, $r_2 = 0.9995$, and $r_3 = 0.9940$.

TABLE 3. Iteration number to achieve convergence with $(x_0, y_0, z_0) = (0.25, 0.15, 0.6)$

	Fixed point of the convergence			
Consensus model	x	y	z	Number of iterations
DeGroot	0.333333	0.333333	0.333333	14
QSO	0.333333	0.333333	0.333333	3
DSQO	0.333333	0.333333	0.333333	5
Geometric QSO [a]	0.333868	0.333333	0.332799	2

[a] $r_1 = 0.9999$, $r_2 = 0.9965$, and $r_3 = 0.9988$.

models, including QSOs, DSQOs, and the Geometric QSO, achieve convergence more rapidly than the DeGroot linear model in scenarios involving both stochastic and doubly stochastic matrices.

The results indicate that achieving nonlinear consensus convergence using QSOs involves a more complex function and requires a greater number of polynomial coefficients compared to the Geometric QSO model. Specifically, for a system with three agents denoted as x, y, and z, the QSO approach necessitates calculating nine polynomial coefficients for each agent. In contrast, the proposed Geometric QSO model, as defined in Eq. (11), requires only three polynomial coefficients per agent, which are $A(G_{r_i})$, $B(G_{r_i})$, and $C(G_{r_i})$ for $i = 1, 2, 3$. This leads to more computational time to reach consensus convergence in QSOs compared to the proposed model.

Even though the number of polynomial coefficient in DSQOs is smaller than in Geometric QSO, one can observe the iteration number to achieve the consensus convergence for both models, where the Geometric QSO nonlinear model with three parameters relatively converges faster than the DSQOs nonlinear model.

The Geometric QSO model demonstrates a notable advantage in its rapid convergence to a fixed point compared to other consensus models. This faster stabilization can be critical in time-sensitive applications, making the model particularly attractive. Although the Geometric QSO does not converge exactly to the center, as achieved by the DeGroot, QSO, and DSQO models, the deviation is negligible in most practical scenarios and does not compromise the utility of the model in solving consensus problems. Moreover, the Geometric QSO requires fewer polynomial coefficients than traditional QSO models, simplifying computations and enhancing efficiency. This trade-off between minor deviations in the fixed point and significantly faster convergence highlights the model's potential for applications demanding both speed and computational simplicity. Future studies could explore hybrid methods or refinements to align the fixed point more closely with the center while preserving the model's efficiency.

CONCLUSION

This study introduces an alternative mathematical model for addressing the consensus problem in MAS through the construction of a Geometric QSO nonlinear model. A thorough theoretical analysis was conducted, followed by a comparative evaluation against three established consensus models: the DeGroot model, QSO model, and DSQO model. The findings reveal that the Geometric QSO model achieves faster convergence than the other models, despite its relatively complex form. However, a notable downside of the Geometric QSO model is that it does not converge exactly to the center, as achieved by the other models. Instead, it converges to a fixed point very close to the center, which is typically sufficient for practical applications. Additionally, the model requires only three parameter values for its probability measure $P_{ij,k}$, simplifying its implementation compared to traditional QSO models. This research contributes significantly to advancing the application of QSO theory, addressing one of the most intricate challenges in nonlinear operator theory, and opens pathways for refining the model to balance its convergence speed with improved accuracy.

ACKNOWLEDGMENTS

We would like to convey our thanks to the Universiti Tunku Abdul Rahman Research Fund (UTARRF) Cycle 2 2023 with project number: IPSR/RMC/UTARRF/2023-C2/S01 for their support.

REFERENCES

1. G. Shi, K. H. Johansson, and Y. Hong, "Reaching an optimal consensus: Dynamical systems that compute intersections of convex sets," IEEE Trans. Autom. Control. **88**, 610–622 (2013).
2. T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, and O. Shochet, "Novel type of phase transition in a system of self-driven particles," Phys. Rev. Lett. **75** (1995).
3. V. Borkar and P. Varaiya, "Adaptive control of markov chains, i: finite parameter set," IEEE Trans. Autom. Control **24**, 953–957 (1979).
4. J. Tsitsiklis, D. Bertsekas, and M. Athans, "Distributed asynchronous deterministic and stochastic gradient optimization algorithms," IEEE Trans. Autom. Control. **31**, 803–812 (1986).
5. M. H. DeGroot, "Reaching a consensus," JJ. Am. Stat. Assoc. **69**, 118–121 (1974).
6. R. L. Berger, "A necessary and sufficient condition for reaching a consensus using degroot's method," J. Am. Stat. Assoc. **76**, 415–418 (1981).
7. M. Cao, A. S. Morse, and B. D. O. Anderson, "Reaching a consensus in a dynamically changing environment: A graphical approach," SIAM J. Control Optim. **47**, 575–600 (2004).
8. J. M. Hendrickx, G. Shi, and K. H. Johansson, "Finite-time consensus using stochastic matrices with positive diagonals," IEEE Trans. Autom. Control. **60**, 1070–1073 (2015).
9. W. Hu, L. Liu, and G. Feng, "Consensus of linear multi-agent systems by distributed event-triggered strategy," IEEE Trans. Cybern. **46**, 148–157 (2016).
10. G. Cui, S. Xu, F. L. Lewis, B. Zhang, and Q. Ma, "Distributed consensus tracking for non-linear multi-agent systems with input saturation: a command filtered backstepping approach," IET Control Theory Appl. **10**, 509–516 (2016).
11. L. Xiao and S. Boyd, "Fast linear iterations for distributed averaging," Syst. Control Lett. **53**, 65–78.
12. P. Kokotović and M. Arcak, "Constructive nonlinear control: a historical perspective," Automatica **37**, 637–662 (2001).
13. A. Roshanzamir, F. Piltan, A. Jahed, S. Namvarchi, N. B. Sulaiman, and I. Nazari, "Research on nonlinear automation for first order delays system," Int. J. Hybrid Inf. Technol. **8**, 313–328 (2015).
14. R. Abdulghafor, S. Turaev, A. Zeki, and A. Abubaker, "Nonlinear convergence algorithm: Structural properties with doubly stochastic quadratic operators for multi-agent systems," J. Artif. Intell. Soft Comput. Res. **8**, 49–61 (2018).
15. R. Abdulghafor, S. Almotairi, H. Almohamedh, S. Turaev, and B. Almutairi, "Nonlinear consensus protocol modified from doubly stochastic quadratic operators in networks of dynamic agents," Symmetry **11** (2019).
16. M. Saburov and K. Saburov, "Reaching a consensus: A discrete nonlinear time-varying case," Int. J. Syst. Sci. **47**, 1–9 (2015).
17. S. Bernstein, "Solution of a mathematical problem connected with the theory of heredity," Ann. Math. Stat. **13**, 53–61 (1942).
18. S. S. Vallander, "On the limit behavior of iteration sequence of certain quadratic transformations," Sov. Math. Dokl. **13**, 123–126 (1972).
19. Y. I. Lyubich, E. Akin, A. Karpov, and D. Vulis, *Mathematical structures in population genetics* (Springer, 1992).
20. R. N. Ganikhodzhaev and F. Shahidi, "Doubly stochastic quadratic operators and birkhoff's problem," Linear Algebra Appl. **432**, 24–35 (2010).
21. N. Ganikhodjaev and N. Z. A. Hamzah, "Geometric quadratic stochastic operator on countable infinite set," in *AIP Conference Proceedings*, Vol. 1643.
22. S. N. Karim, N. Z. A. Hamzah, and N. Ganikhodjaev, "On nonhomogeneous geometric quadratic stochastic operators," Turk. J. Math. **46**, 1397–1407 (2022).
23. S. N. Karim, N. Z. A. Hamzah, and N. Ganikhodjaev, "On the dynamics of geometric quadratic stochastic operator generated by 2-partition on countable state space," Malays. J. Math. Sci. **16**, 727–737 (2022).
24. R. Abdulghafor, S. Turaev, A. Abubakar, and A. Zeki, "The extreme doubly stochastic quadratic operators on two dimensional simplex," in *2015 4th International Conference on Advanced Computer Science Applications and Technologies (ACSAT)* (2015) pp. 192–197.