SUB-EXACT SEQUENCE ON HILBERT SPACE

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Introduction

- The notion of the sub-exact sequence is a generalization of exact sequence in algebra especially on a module.
- A module over a ring R is a generalization of the notion of vector space over a field F.
- Refers to a special vector space over field F when we have a complete inner product space, it is called a Hilbert space.
- In this paper, we introduce the new point of view that we could construct the sub-exact sequence on Hilbert space. Furthermore, we investigate the basic properties of the sub-exact sequence.

Sub-Exact Sequence

- The sub-exact sequence is a concept that is applied in the module, which is a generalization of vector space.
- Recall an inner product space V over F is a vector space V together with an inner product $\langle , \rangle : V \times V \to F$.
- When we have a complete inner product space, it is called a Hilbert space.
- A space is complete if every Cauchy sequence converges.

Sub-Exact Sequence

- **Definition 2.1** *Let* K, L, M *be R-modules and X be a submodule of* L*. Then the triple of* (K, L, M) *is said to be X-sub-exact at* L *if there exist R-homomorphisms* f *and* g *such that the sequence of R-modules and R-homomorphisms*

$$K \xrightarrow{f} X \xrightarrow{g} M$$

is exact. The sequence is said to be exact if $Im(f) = Ker(g)$*.*

Sub-Exact Sequence

- **Definition 2.2** *Let* H_1, H_2, H_3 *be Hilbert space over field F and X be a subspace of* H_2 (*X need not a Hilbert space*). *Then the triple of* (H_1, H_2, H_3) *is said to be an X-sub-exact at* H_2 *if there exist linear transformation* α *and* β *such that the sequence of Hilbert space and homomorphisms*

$$H_1 \xrightarrow{\alpha} X \xrightarrow{\beta} H_3$$

is exact.

Sub-Exact Sequence

Example 2.1 Let $H_1 = \mathbb{R}^2$, $H_2 = \mathbb{R}^3$, and $H_3 = \mathbb{R}^n$ be Hilbert spaces with an inner product is defined by

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$$

and metric induced by this inner product is

$$d(x,y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{\frac{1}{2}}.$$

Let $\mathbb{R}$ is a subset of $\mathbb{R}^3$ Hilbert space, then the triple $(\mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^n)$ is $\mathbb{R}$-*sub-exact* at $X = \{(r, 0,0) | r \in \mathbb{R}\} \subset \mathbb{R}^3$ since there are homomorphisms α and β as follow

$$\alpha\colon \mathbb{R}^2 \to X$$
$$(x, y) \longmapsto x$$

and

$$\beta\colon X \longrightarrow \mathbb{R}^n$$
$$x \longmapsto (0,0,\dots,0)$$

It is clear that $Im\ (\alpha) = Ker\ (\beta)$. Hence, the sequence $\mathbb{R}^2 \xrightarrow{\alpha} \mathbb{R} \xrightarrow{\beta} \mathbb{R}^n$ is exact. Then the triple of $(\mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^n)$ is $\mathbb{R}$-sub-exact at $\mathbb{R}^3$.

Results

- **Proposition 2.1** *Let* H_1, H_2, H_3 *be Hilbert spaces and* V, S *be subspaces of* H_2, *where* $S \subset V$. *If* $V \in \sigma(H_1, H_2, H_3)$, *and* S *is a direct summand of* V *then* $S \in \sigma(H_1, H_2, H_3)$.

Proof:

Since $V \in \sigma(H_1, H_2, H_3)$, there are homomorphisms α and β such that the sequence

$$H_1 \xrightarrow{\alpha} V \xrightarrow{\beta} H_3$$

is exact.

Since S is a direct summand of V, there exist, T, a subspace of V such that $V = S \oplus T$. Hence, for every $v \in V, v = s + t$ for some $s \in S$ and $t \in T$. Then we define a homomorphism

$$p: V = S \oplus T \to S$$

where $p(v) = p(s + t) = s \in S$.

Now, let $g = \beta|_S$. We will show that $Ker\ (g) = Im\ (f)$. Let $x \in Ker\ g \subseteq S$. Then $g(x) = \beta(x) = 0$. Hence, $x \in Ker(\beta)$. Since $Im\ \alpha = Ker\ (\beta)$, there is $k \in H_1$ such that $\alpha(k) = x$. Then $f(k) = (p \circ \alpha)(k) = p\big(\alpha(k)\big) = p(x) = x$. This implies, $Ker\ (g) \subset Im\ (f)$.

Now, let $x \in Im\ (f) \subseteq S$. We have $k \in H_1$ such that $f(k) = x$. Then $x = f(k) = p \circ \alpha(k) = \alpha(k)$. Hence, $x \in Im\ (\alpha) = Ker(\ \beta)$. Therefore, $\beta(x) = 0$. Since $x \in S$, $g(x) = \beta(x) = 0$. So that $x \in Ker\ (g)$. Hence, $Im\ (f) \subset Ker\ (g)$.

We have $Im\ (f) = Ker\ (g)$. So, the sequence $H_1 \xrightarrow{f} S \xrightarrow{g} H_3$ is exact. Therefore, $S \in \sigma(H_1, H_2, H_3)$. ■

- **Definition 2.3** *Let* V *and* W *be inner product spaces and let* $\pi \in \mathcal{L}(V, W)$, *then* π *is an isometry if it preserves the inner product, that is, if*

$$\langle \pi u, \pi v \rangle = \langle u, v \rangle$$

for all $u, v \in V$.

- **Proposition 2.2** *Let* H_1, H_2 *be two Hilbert spaces and* W *be a subspace of* H_2*. If there is an isometric isomorphism from* H_1 *to* W *then the triple* $(H_1, H_2, \mathbf{0})$ *and* $(\mathbf{0}, H_1, H_2)$ *are* W*-sub-exact sequences.*

- **Proof.** Let $H_1 \xrightarrow{\pi} W \to \mathbf{0}$ and let $\mathbf{0} \to H_1 \xrightarrow{\tau} W$ where π and τ are isometric isomorphism. From Definition 3.5 in [1], the sequence $H_1 \to W \to \mathbf{0}$ is exact since there is π a surjective mapping from V to W. And also the sequence $\mathbf{0} \to H_1 \to W$ is exact since there is τ an injective mapping from H_1 to W. Since W is a subspace of H_2, from Definition 2.2, the triple of $(H_1, H_2, \mathbf{0})$ is said to be a W-sub-exact at H_2.■

- **Proposition 2.3** *Let H_1, H_2 be Hilbert spaces, W subspace of H_2, and δ_1, δ_2 be two metrics in H_1 and H_2, respectively. If τ is an isometric isomorphism from H_1 to W then there is an injective isometric ω from H_1 to H_2 such that*

$$\delta_1(v_1, v_2) = \delta_2\big(\omega(v_1), \omega(v_2)\big)$$

Proof:

We can define a linear transformation ω from H_1 to H_2 by $\omega = i \circ \tau$. Since i and τ are injective mappings then ω is also injective.

Furthermore, we will show that ω is isometric. By the assumption that τ is an isometric isomorphism from H_1 to *W*, we have

$\delta_1(v_1, v_2) = \delta_2\big(\tau(v_1), \tau(v_2)\big)$.

Since *W* is a subspace of H_2, we get

$\delta_1(v_1, v_2) = \delta_2\big((i \circ \tau)(v_1), (i \circ \tau)(v_2)\big) = \delta_2\big(\omega(v_1), \omega(v_2)\big)$.

Then, ω is an injective isometric from H_1 to H_2.■

Conclusion

- If there are three Hilbert spaces over field F, namely H_1, H_2, H_3 then the collection of all subspaces X of H_2 that the triple (H_1, H_2, H_3) is X-sub-exact denoted by $\sigma(H_1, H_2, H_3)$. When we have two any subspaces of H_2, namely V and S, and V is an element of the collection of all subspaces that sub-exact at H_2 then S is also sub-exact sequence at H_2, where S is a direct summand of V. Furthermore, we have two properties of isometric isomorphism sub-exact sequence on Hilbert space.
- If there are three Hilbert spaces where one of them is null space, and there is an isometric isomorphism from H_1 to W subspace of H_2, then the triple $(H_1, H_2, \mathbf{0})$ and $(\mathbf{0}, H_1, H_2)$ are W-sub-exact sequence at H_2.
- If any two Hilbert spaces and there is an isometric isomorphism from the Hilbert space to its subspace, namely τ from H_1 to W subspace of H_2 then we will find an injective isometric ω among two Hilbert spaces from H_1 to H_2.

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