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RESEARCH ARTICLE

Scattering of the vector soliton in coupled nonlinear Schrödinger equation with *Gaussian* potential

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Abstract

Nonlinear Schrodinger equation (NLSE) is the fundamental equation which describes the wave field envelope dynamics in a nonlinear and dispersive medium. However, if the fields have many components, one should consider the Coupled Nonlinear Schrodinger equation (CNLSE). We considered the interactions of orthogonally polarized and equal-amplitude vector solitons with two polarization directions. In this paper, we focused on the effect of Gaussian potential on the scattering of the vector soliton in CNLSE. The scattering process was investigated by the variational approximation method and direct numerical solution of CNLSE. Analytically, we analyzed the dynamics of the width and center of mass position of a soliton by the variational approximation method. Soliton may be reflected from each other or transmitted through or trapped. Initially, uncoupled solitons may form the coupled state if the kinetic energy of solitons less than the potential of attractive interaction between solitons but when its' velocity above the critical velocity, the soliton will pass through each other easily. Meanwhile, a direct numerical simulation of CNLSE had been run to check the accuracy of the approximation. The result of the variational model gives a slightly similar pattern with direct numerical simulation of CNLSE by fixing the parameters for both solutions with the same value. The interaction of the vector soliton with Gaussian potential depends on the initial velocity and amplitude of the soliton and the strength of the external potential.

Keywords: Vector soliton, coupled nonlinear schrodinger equation, gaussian potential, variational method, numerical method

INTRODUCTION

The Nonlinear Schrodinger equation (NLSE) is among the prominent equations in nonlinear system which describes the propagation of the wave through a nonlinear medium as shown in Eq. (1). It is also applied in nonlinear quantum field theory, condensed matter and plasma physics, fluid mechanics, and star formation (Tian *et al.,* 2005). Derived by Benney and Newell (1967), the same equation also rules light pulse propagation in optical fibers as proven by Hasegawa and Tappert in 1973.

$$i\Psi_t + \Psi_{xx} + 2|\Psi|^2 \Psi = 0 \qquad (1)$$

Coupled Nonlinear Schrodinger equation (CNLSE) comes into the picture when subject of collision of vector solitons arises. Based on previous study, it was shown that the collisions are generally inelastic (Ueda and Kath*.,* 1990). When collision of vector solitons happened, there is a possibility for a phase shift but remain unchanged in term of velocity and shape as the velocity is at constant (Goodman *et al.,* 2005). The CNLSE are as follows

$$iA_t + A_{xx} + (|A|^2 + \beta |B|^2)A = 0 \qquad (2)$$

$$iB_t + B_{xx} + (|B|^2 + \beta |A|^2)B = 0 \qquad (3)$$

C.R. Menyuk first derived the nonlinear pulse propagation equation in single-mode optical fiber (SMF) under weak birefringence in 1987. Then, he described vector soliton as two different solitons having different polarization component. Referring to Eq. (2) and Eq. (3), the vector soliton A is initially polarized to the vector soliton B differently, and it impacted each other upon collision. The term β is called as the cross-phase modulation coefficient (Yang, 2010). The CNLSE is reduced to become NLSE if the system is integrable and thus passes each other without any changes in amplitude, velocity, or polarization. The phenomena of CNLSE is illustrated in Fig. 1.

The formulation of the problems in terms of the differential equations plays a crucial role in science and engineering. In physics, nearly all basic laws have been presented as differential equations for time and space evolution of physical quantities. For examples, the Newton equations of classical mechanics, the Schrodinger equation in quantum mechanics, and the Einstein equations of general relativity. While in early stages emphasis was given to analytical solutions of linear equations. Recently, nonlinear problems have been widely utilized in fundamental and applied physics, which are described by a system of nonlinear partial or ordinary differential equations, in full complexity (Chai *et al.,* 2015). It was found that some nonlinear equations with significant applications in physics have property of exact integrability, and consequently their solutions can be found and described in details (Pitaevskii *et al.,* 2003). The mathematical development of the theory of integrable nonlinear equations was essential for many fields of physics. It was found that these kinds of equations may have localized solutions (solitons), and the interaction of solitons looks like particle interaction, so after collision they can preserve their shape and velocity (Umarov *et al.,* 2017).

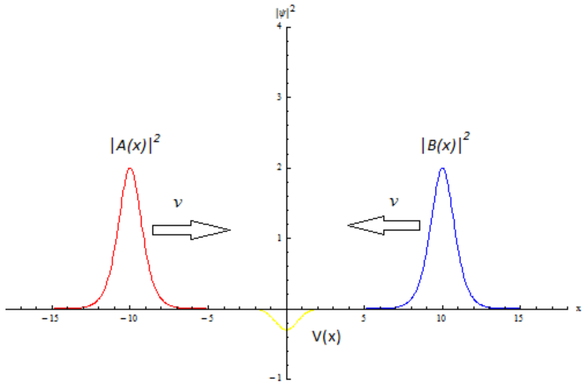


Fig. 1 Two solitons moving in different directions and scattering on the external potential V(x).

However, not all nonlinear equations are exactly integrable. It may become nonintegrable if one considers the physical effects such as collision and perturbation effect. Nonintegrable nonlinear equations may still have soliton like solutions, but their interactions in general become more complex. Furthermore, an exact analytical solution of a nonintegrable equations can be found only for some very limited cases, so one has to apply approximate or numerical methods (Molomed, 2008).

Therefore, this work examined few problems related to interaction and dynamics of vector solitons in the physical systems theoretically described by CNLSE. In this present paper, we focused on two physical objects namely the matter-wave soliton in Bose-Einstein condensate and the optical soliton in fibers. The dynamics of these physical systems can be considered based on the nonintegrable modifications of the one-dimensional Nonlinear Schrodinger equation

The plan of the present paper is as follow. In section 2 we briefly describe the model and the main equation for scattering of vector soliton with external potential and obtain the perturbated Coupled-NLSE. In section 3, the variational approximation method is described followed by three representative set of numerical result for ordinary differential equation (ODE). The types of soliton transmission, reflection and trapping behaviors are also discussed in in section 4. In section 5 we compare the outcome between partial differential equation (PDE) and ODE results by using same range of parameter. Then, the conclusion of this study is discussed in section 6.

DESCRIPTION OF THE MODEL

In this paper, we implemented the same method to derive similar qualitative features in Goodman and Haberman's ODE model of vector solitons collision in CNLSE, and to explain the fractal structure of resonance windows. Here, we extend this approach including the external potential acting on the system. We consider the main equation of our model, the Coupled-Nonlinear Schrodinger Equation (CNLSE) with perturbation

$$iA_t + A_{xx} + (|A|^2 + \beta |B|^2 - V(x))A = 0 \tag{4}$$

$$iB_t + B_{xx} + (|B|^2 + \beta |A|^2 - V(x))B = 0 \tag{5}$$

where *A* and *B* is the wave function while β is cross-phase modulation coefficient and *V(x)* is the external potential. In our case, we consider the interaction of two orthogonally polarized and equal-amplitude vector solitons with localized potential well. The initial condition for this collision may be written as

$$A(x,0) = \sqrt{2}\,\mathrm{Sech}\left(x + \frac{1}{2}X_0\right)\exp[\frac{1}{4}iv_0 x] \tag{6}$$

$$B(x,0) = \sqrt{2}\,\mathrm{Sech}\left(x - \frac{1}{2}X_0\right)\exp[-\frac{1}{4}iv_0 x] \tag{7}$$

where X_0 is the initial center of mass position and v_0 is the relative velocity of the two initial solitons. In other words, the left soliton (6) is initially polarized in the A component and the right soliton (7) initially polarized in the B component. The value of X_0 does not affect the result of collision as long as it is large enough. In fact, both vector solitons are related by the symmetry *B(x,t) = A(-x,t)* due to initial conditions. Due to this, we only showed the contours of A component, as the countors of B component could be demonstrated by a simple mirror reflection of the A contours about position $x = 0$. In absence of external potential $(\mathrm{V(x)} = 0)$, together with $\beta = 0$ or $\beta = 1$, the CNLSE will be reduced to become NLSE, further this system is integrable. These two NLS solitons will pass through each other without any changes in amplitude, velocity and polarization as demonstrated in Fig. 2.

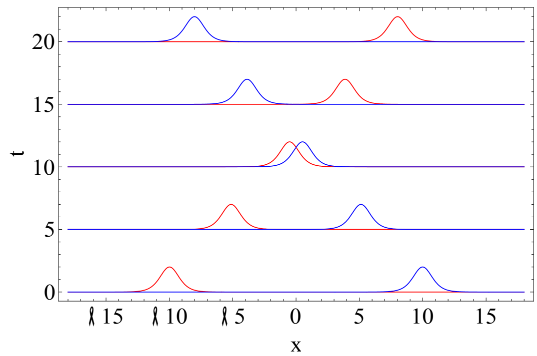


Fig. 2 Time evolution of vector solitons for $\mathrm{V(x)} = 0$ and $\beta = 0$.

The system will be non-integrable if $\beta \neq 0$ and the amplitude, velocity and polarization of initial soliton also change after the collision (Malomed, 2017). In this work, we consider the interaction between two solitons are weak by choosing the small value of β , hence the shape of solitons remains close to its original but the width and amplitude may oscillate during the collisions. Here we varied the velocity of vector solitons in order to study its influences on the nonlinear scattering of vector solitons with localized Gaussian potential well, Eq. (8).

$$V(x) = -V_p \exp[-\delta x^2] \tag{8}$$

where δ and V_p is the width and the strength of the potential.

VARIATIONAL APPROXIMATION APPROACH

To understand better about the scattering process of the vector solitons with external potential, we analytically study this problem in this section. The CNLSE is solved approximately by using variational approximation method. This method is used to find an approximate inference based on the parameters of the solitons in the existing complex models. It works by simplifying the PDE to the ODE which easier to be used later. This method is relying on the choice of ansatz, which should be substituted to the Lagrangian for further evaluation and derivation of Lagrangian equations. The appropriate ansatz is chosen based on the known soliton solutions for NLSE, which, as previous studies show works well for more general NLS type equations.

The variational approximation method can be used to get the robust properties of CNLSE solution. However, the accuracy of this approximation is crucially depending on the chosen ansatz (an assumed wave function of the system which is made in order to ease solution of an equation). The results from this method may become invalid if energy radiation is significant, therefore it will be neglected.

In this paper, a Lagrangian density was first derived from the CNLSE before we substituted the ansatz in the equation.

$$\mathrm{L} = i(AA_t^* - A_t A^*) + i(BB_t^* - B_t A^*) + (2|A_x|^2 - |A|^4) + (2|B_x|^2 - |B|^4) - 2\beta|A|^2|B|^2 + 2V(x)|A|^2 + 2V(x)|B|^2. \quad (9)$$

We also have considered two appropriate choices of trial functions which are the secant ansatz (Tan *et al.,* 2001)

$$A(x,t) = \sqrt{2}a\,\mathrm{Sech}\left(\frac{x+X/2}{\omega}\right)\exp\left[i\left(\frac{1}{4}v(x+X/2)+\frac{b}{2\omega}(x+X/2)^2+\sigma\right)\right] \quad (10)$$

$$B(x,t) = \sqrt{2}a\,\mathrm{Sech}\left(\frac{x-X/2}{\omega}\right)\exp\left[i\left(-\frac{1}{4}v(x-X/2)+\frac{b}{2\omega}(x-X/2)^2+\sigma\right)\right] \quad (11)$$

where *a, X, w, v, b* and are amplitude, center of mass, width, velocity, chirp parameter, and initial phase of soliton, respectively. We substituted the chosen ansatz into the Lagrangian density accordingly and then calculate the effective Lagrangian with spatial integration of the Lagrangian density, $L = \int_{-\infty}^{\infty} \mathrm{L}\,dx$. Here it is important to take note of that the second last and last term in effective Lagragian Eq. (9) cannot be solved analytically for arbitrary potential *V(x).* Thus, we present analytical approximation for the integral related with potential and consider it approximated by Gaussian function,

$$\int_{-\infty}^{\infty} \mathrm{Sech}\left[\frac{x+\frac{X}{2}}{\omega}\right]dx = \int_{-\infty}^{\infty} \exp\left[-\frac{\left(x+\frac{X}{2}\right)}{\omega^2 c^2}\right]dx \quad (12)$$

where $c = 2/\sqrt{\pi}$. Thus, it produced the total averaged Lagrangian,

$$L = N\left[2vX_t + \frac{2}{3}\pi^2 b_t\omega - \frac{2}{3}\pi^2 b\omega_t + 8\sigma_t + \frac{16}{3\omega^2} + v^2 + \frac{4}{3}b^2\pi^2 - \frac{32N}{3\omega} - \frac{32N\beta}{\omega^2}\left(\frac{X\,\mathrm{Cosh}\left[\frac{X}{\omega}\right] - \omega\,\mathrm{Sinh}\left[\frac{X}{\omega}\right]}{\mathrm{Sinh}^3\left[\frac{X}{\omega}\right]}\right) - \frac{4\sqrt{\pi}cV_p}{\sqrt{\delta\omega^2c^2+1}}\exp\left(-\frac{\delta X^2}{4(\delta\omega^2c^2+1)}\right)\right] \quad (13)$$

The norm of the wave function of vector soliton $N = wa^2$ is a conserved quantity and proportional to the number of atoms in the condensate region. Using Euler-Lagrange equations, $d/dt(\delta L/\delta q_t)-(\delta L/\delta q_t)=0$. We derived all of the parameters, where q are time dependent collective coordinates *b,X, w ,σ, N and v*. It is decoupled from other equation and can be dropped by using the relations of $dX/dt = v$ and $dw/dt = b$. What remain is a set of coupled equation for the center of mass position and width of the soliton.

$$\frac{d^2X}{dt^2} = \frac{16N\beta}{\omega^3}\left[\left(X - 3X\,\mathrm{Coth}^2\left[\frac{X}{\omega}\right] + 3\omega\,\mathrm{Coth}\left[\frac{X}{\omega}\right]\right)\mathrm{Csch}^2\left[\frac{X}{\omega}\right]\right] - \frac{2X\delta V_p}{\sqrt[3]{\delta\omega^2c^2+1}}\exp\left(-\frac{\delta X^2}{4(\delta\omega^2c^2+1)}\right) \quad (14)$$

$$\frac{d^2\omega}{dt^2} = \frac{16}{\pi^2\omega^2}\left[\frac{1}{\omega} - N - \frac{3N\beta}{\omega^2}\left(X^2 - 3X^2\,\mathrm{Coth}^2\left[\frac{X}{\omega}\right] + 4X\omega\,\mathrm{Coth}\left[\frac{X}{\omega}\right] - \omega^2\right)\mathrm{Csch}^2\left[\frac{X}{\omega}\right]\right] - \frac{12\delta\omega c^2V_p}{\pi^2\sqrt[3]{\delta\omega^2c^2+1}}\exp\left(-\frac{\delta X^2}{4(\delta\omega^2c^2+1)}\right)\left[1 - \frac{\delta X^2}{2(\delta\omega^2c^2+1)}\right] \quad (15)$$

The simulation of the reduced ODE model Eqs. (14)-(15) reveal that this model captures the main features of soliton scattering by external potential.

ODE SIMULATION OF THE INITIAL VALUE PROBLEM

We presently display a set of numerical outcomes for the ODE which are the Eqs. (14) – (15) with initial conditions,

$$X(0) = -10, X'(0) = V_0, \omega(0) = 1, \omega'(0) = 0, N = 1, \beta = 0.2 \quad (16)$$

It has been discovered that the soliton acts like classical particle where it moves freely alongside the steady width parameter of soliton (Abdullaev *et al.,* 2000). Soliton might be transmitted, trapped, or reflected from the potential depending on the velocity of the vector soliton. When the soliton reaches to the potential, both velocity and width of soliton are be affected by the perturbation. It is seen that the numerical result of the Eqs. (9) – (10) are shown in the Fig. 3,4 and 5.

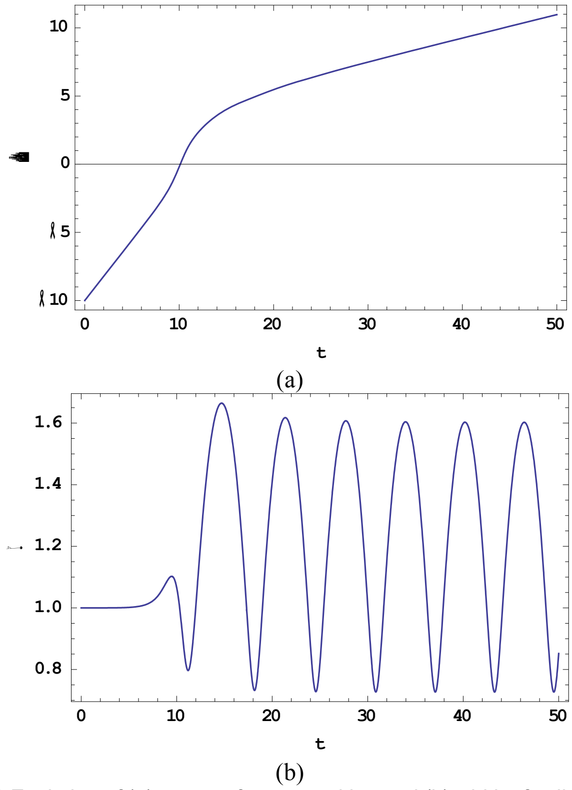
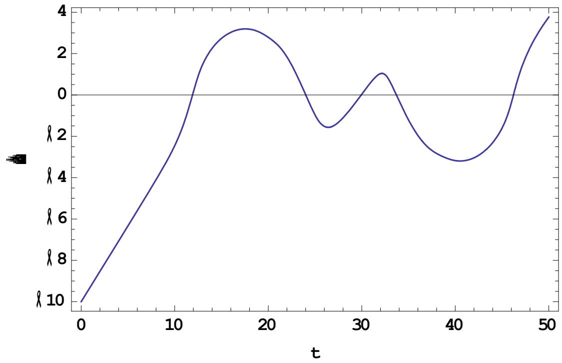
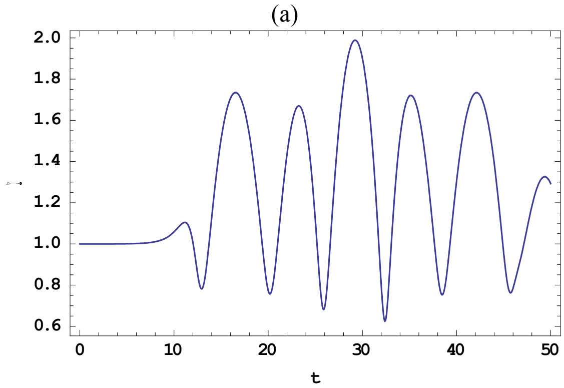


Fig. 3 Evolution of (a) center of mass position and (b) width of solitons in the presence of a Gaussian potential well at $V_0 = 0.885$.

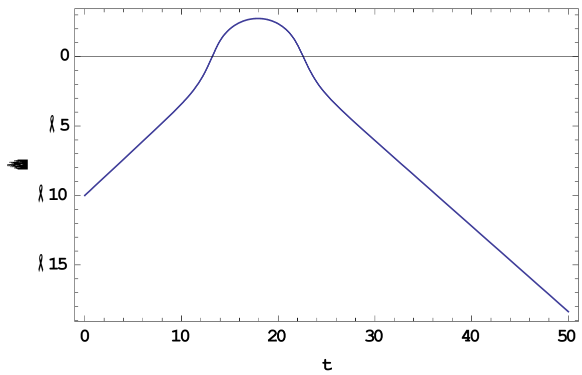


(a)

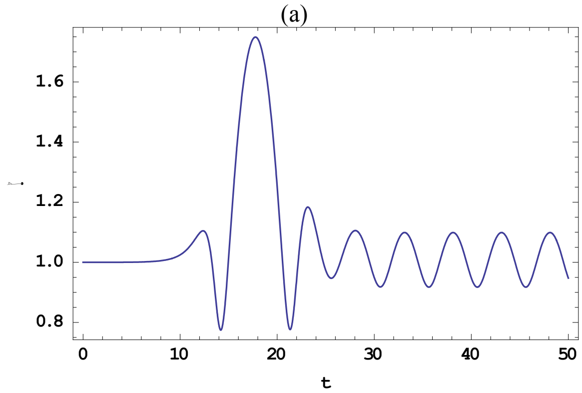


(b)

Fig. 4 Evolution of (a) center of mass position and (b) width of solitons in the presence of a Gaussian potential well at $V_0 = 0.733$.



(a)



(b)

Fig. 5 Evolution of (a) center of mass position and (b) width of solitons in the presence of a Gaussian potential well at $V_0 = 0.650$.

Fig. 3 shows the vector solitons are transmitted each other particularly during the blue line cross the horizontal x-axis indicating the vector solitons penetrating each other. The vector solitons are 'trapped' in each other during the collision as in Fig. 4. While in Fig. 5, the vector solitons reflected each other after the collision and return to its original position. The result is quite similar with previous cases (Aklan *et al.*, 2015) where when its velocity above the critical velocity (v_c=0.083), the soliton passes through the potential. While for the soliton with low energy, it will be reflected or trapped by the potential. In addition, the vector soliton moves freely with constant velocity when it is far from the potential, but it will be affected when it is approaching the potential. Because of that their amplitude and width also oscillate after the scattering. We can see that the oscillation decay over the time due to the energy loss to surrounding.

NUMERICAL RESULT

The variational approximation technique gave the approximate results and supported by some assumptions. While direct numerical solutions of the governing equations are performed to examine the accuracy of the approximate calculations. In order to check the dynamics of vector solitons, split-step Fast Fourier Transform (FFT) procedure is applied to resolve the CNLSE numerically; this technique relies on the spectral ways. Numerically, we monitor the initial velocity of the vector soliton by using Eq. (18) and compare with dynamic equation of variational approximation Eqs. (14) – (15),

$$X(t_q) = \left. \frac{\int_{-\infty}^{\infty} x\,|\psi(x,t_q)|^2\,dx}{\int_{-\infty}^{\infty} |\psi(x,t_q)|^2\,dx} \right|_{\substack{q=i,f \\ \Psi=A,B}} \tag{17}$$

$$v = \frac{X(t_f) - X(t_i)}{t_f - t_i} \tag{18}$$

where *v, X,* ψ, t_i *and* t_f are velocity, position, wave function, initial time, and final time of the vector soliton. Firstly, we simulated Eq. (4) and Eq. (5) extensively with initial condition Eq. (6) and Eq. (7) using the initial velocity V_0 as variable. We take a large value for the initial position of vector soliton which is X_0 = 10. We use second order split step method as our numerical scheme. The spatial domain was taken as [-90,90] and 1024 grid point were used. The time step was selected as 0.01. To reduce radiation which will feed back to the system through periodic boundary condition, a boundary absorber has been applied to both end of the interval. To minimize numerical errors, we have selectively run our simulation with wide range of x interval, large scale of grid point and smaller time steps.

We determine the exit velocity V_∞ of a transmissional or reflectional collision as the difference between these two velocities of vector soliton. We run the programme for a long period of time and track the position of the soliton, If these two position still remain very close to each other, we decide that the two solitons are trapped in each other and merely assign V_∞ as zero. For reflection collision, we define V_∞ as negative. While for transmission collision, V_∞ is positive. In Fig. 6 and 7, we demonstrate the fractal structure in the exit velocity graph for both ODE model and direct numerical of Coupled NLSE (PDE model) .

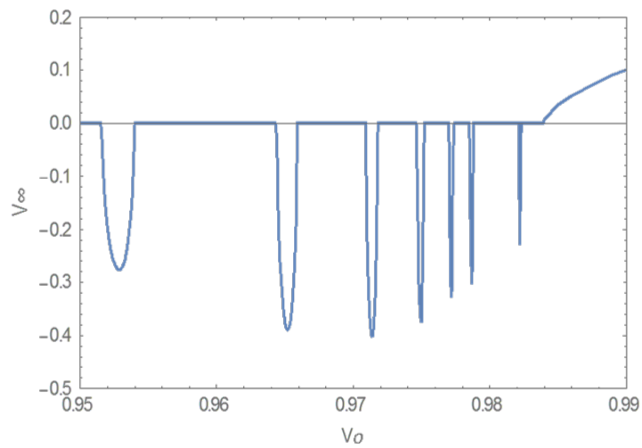


Fig. 6 The exit velocity versus collision velocity for direct numerical of Coupled NLSE Eq. (18) at $\beta = 0.2$.

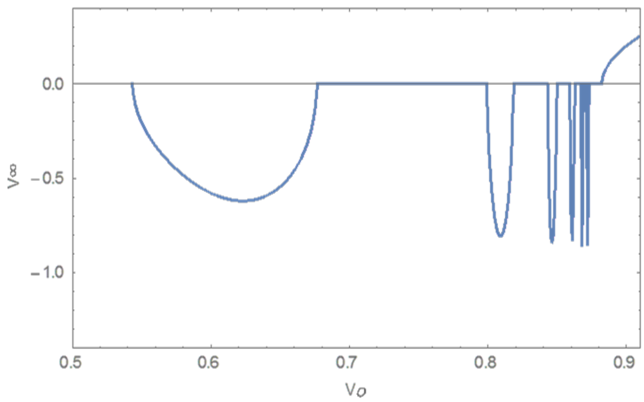


Fig. 7 The exit velocity versus collision velocity for ODE model Eqs. (14)-(15) at $\beta = 0.2$.

There are key similarities and differences between the exit velocity for ODE model and full PDE simulation. The simulations of ODE model show that this model quantitatively explains the main component of the collisions in the PDE (4)-(5). For example, the simulation result of the ODE model at $\beta = 0.2$ display almost the same pattern of graph for PDE as in Fig. 6 and Fig. 7. In addition, the critical velocity for the ODE's, v_c=0.883 is quantitatively close to the value v_c=0.984 found in PDE's, as stated in the description of the figure. These results are very encouraging and further completes the analysis of the model. However, there are some differences yielded between these two systems.

First, the different value of the critical velocity of PDE simulations and ODE approximations. This is because of the simplification of the main equation of the model in ODE. Second, the number of reflections happened across time were quite different in PDE and ODE model. The Fig. 6 and 7 clearly showed that some vector solitons reflection collision was missing from one another. The reasons for these differences are mostly due to the omission of energy radiation in the ODE model. This radiation mostly produced when they pass through each other. It dissipates the soliton motion hence escaping becoming more difficult. This dissipation mechanism is not present in the ODE model. As a result of the dissipation, in PDE solution the exit velocity of the reflected solution is much smaller than the input velocity, However, the exit velocity exactly matched the input velocity in the ODE solution. This radiation loss was not considered in this variational method. The result of this method may become invalid, if the radiation is significant in the underlying problem (Yang, 2010).

Differences between ODE and PDE model are mostly quantitative. But the ODE model has clearly captured the main scattering structure and dynamic of PDE model qualitatively.

CONCLUSION AND FUTURE WORK

In this paper, we studied the scattering of vector solitons of Coupled NLSE in the presence of external Gaussian potential. Approximation method is used to describe the result and is compared with the direct numerical solution to observe how accurate the approximation approach can be. The Lagrangian of CNLSE is derived before optimizing the equation by using Euler-Lagrange equation to get the evolution of soliton parameters. The presented results of CNLSE by numerical simulation show that variational approximation well describes the vector solitons behaviors. Direct simulation of CNLSE has already covered all the results needed but can be time consuming to run the simulation, thus variational approach can give faster results and also it gives us the insight to the physics of the vector solitons existence and dynamics.

We hope that this simple simulation will help us to develop intuition and to solve more complicated problems on vector solitons scattering in the future. This research can be extended on vector solitons interaction with different kinds of external potential like barrier, steps, and delta and also other form of generalized CNLSE. This result will serve as guidelines for possible future experiments with matter-wave solitons and optical solitons including their practical applications.

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