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On the uniform summability of the Fourier-Laplace series on the sphere

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Abstract. Convergence problems has been the focus of interest for researchers that are working in the fields of spectral theory. In the current research we investigate issues relating to the summability of the Fourier-Laplace series on the unit sphere. The necessary conditions which are required to obtain good estimation for summability of the Fourier-Laplace series investigated. This research will also provide new and sufficient conditions in the form of theorems and lemmas which will validate the uniform summability of the Fourier-Laplace series on the sphere.

1. Introduction

Let S^N is N dimensional sphere in R^{N+1} :

$$S^N = \{x = (x_1, x_2, ..., X_{N+1}) \in R^{N+1} : \sum_{n=1}^{N+1} x_n^2 = 1\}$$

For any two point x and y in S^N by $\gamma = \gamma(x, y)$ denote spherical distance between these two points which is radial value of an angle between vectors x and y . It is clear that $\gamma \leq \pi$.

Denote by Δ_s be Laplace-Beltrami operator on S^N which has the following expression in the spherical coordinates $x = (\xi_1, \xi_2, ..., \xi_{N-1}, \zeta)$ as:

$$\Delta_s = \frac{1}{\sin^{N-1}\xi_1}\frac{\partial}{\partial\xi_1}\Big(\sin^{N-1}\xi_1\frac{\partial}{\partial\xi_1}\Big) + \frac{1}{\sin^2\xi_1\sin^{N-2}\xi_2}\frac{\partial}{\partial\xi_2}\Big(\sin^{N-2}\xi_2\frac{\partial}{\partial\xi_2}\Big) + \ .\ .\ .\ .+$$

$$+\frac{1}{\sin^2\xi_1\sin^2\xi_2\ .\ .\ .\sin^2\xi_{N-1}}\frac{\partial^2}{\partial\zeta^2}.$$

Consider this operator as a formal differential operator with domain of definition $C^\infty(S^N)$. It is a symmetric, non negative and essentially selfadjoint. Thus its closure $\overline{-\Delta_s}$ is a selfadjoint operator in $L_2(S^N)$. Its eigenfunctions Y^k are known as spherical harmonics and they are complete and orthogonal system in $L_2(S^N)$.

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For the any function $f \in L_2(S^N)$ its Fourier series by spherical harmonics $\{Y_j^k\}\Big|_{j=1}^{a_k}$ is called Fourier-Laplace series on sphere:

$$f(x) = \sum_{k=0}^{\infty}\sum_{j=1}^{a_k} f_{k,j} Y_j^k(x), \tag{1}$$

where $f_{k,j} = \int_{S^N} f(y)Y_j^k(y)d\sigma(y)$, and $a_k = \frac{(N+k)!}{N!k!} - \frac{(N+k-2)!}{N!(k-2)!}$ is a frequency of the corresponding eigenvalues $\lambda_k = k(k+N-1)$. Equality (1) should be understood in the sense $L_2(S^N)$.

A partial sum of the series (1) can be written as follows

$$E_n f(x) = \int_{S^N} f(y)\Theta(x,y,n)d\sigma(y),$$

where $\Theta(x,y,n)$ is called a spectral kernel and has a form:

$$\Theta(x,y,n) = \sum_{k=0}^{n}\sum_{j=1}^{a_k} Y_j^k(x)Y_j^k(y). \tag{2}$$

Equation (1) can be unestood in the sense other than L_2 topology. But in this case application of the summation methods of the partial sum (regularization of the partial sum) is required depending on the smoothnes of the function. Traditionally summation of the Fourier-Laplace series carried of by the Cesaro means [4]. In [4] Kogbetliantz obtained asymptotic representations of the Cesaro means of the spectral kernel of the Fourier-Laplace series.

In the present paper we consider the Riesz method of summation [1]. The Riesz means of the partial sums of the series (1) also has an integral form:

$$E_n^\alpha f(x) = \int_{S^N} f(x)\Theta^\alpha(x,y,n)dy,$$

where the kernel $\Theta^\alpha(x,y,n)$, is the Riesz means of the spectral kernel (2) and has the following asymptotic formula (see [10]):

if $\left|\frac{\pi}{2} - \gamma(x,y)\right| < \frac{\pi n}{2(n+1)}, n \to \infty$, then

$$\Theta^\alpha(x,y,n) = n^{\frac{N-1}{2}-\alpha}(N-1)\frac{\sin\left[(n+\frac{N}{2}+\frac{\alpha}{2})\gamma - \pi\frac{(N-1+2\alpha)}{4}\right]}{(2\sin\gamma)^{\frac{N-1}{2}}\left(2\sin\frac{\gamma}{2}\right)^{1+\alpha}}$$
$$+n^{\frac{N-3}{2}-\alpha}\frac{\eta_n(\gamma)}{(\sin\gamma)^{\frac{N+1}{2}}\left(\sin\frac{\gamma}{2}\right)^{1+\alpha}} + \frac{\varepsilon_n(\gamma)}{(n+1)\left(\sin\frac{\gamma}{2}\right)^{1+N}} \tag{3}$$

where $|\eta_n(\gamma)| < C,\ |\varepsilon_n(\gamma)| < C$;

if $0 < \gamma_0 \le \gamma \le \pi$

$$|\Theta^\alpha(x,y,n)| \le C_4 n^{N-1-\alpha} \tag{4}$$

if $0 \le \gamma \le \pi$

$$|\Theta^\alpha(x,y,n)| \le C_5 n^N. \tag{5}$$

In the present paper we will study the problems of the uniform summability by the Riesz means of the Fourier-Laplace series in the Nikolskii spaces $H_p^a(S^N)$ [13]. These questions for the Cesaro means studied by A.K.Pulatov [5].

2. Main theorem

For any domain $\Sigma \subset S^N$ by Σ^* denote its diametrically opposite: $\Sigma^* = \{x^* \in S^N :$ *there exist* $x \in \Sigma$ *such that* $\gamma(x, x^*) = \pi.$ $\}$

Theorem 2.1 *Let* $f \in H_p^a(S^N),\ p \geq 1,\ a > 0.$

(i) *If* $pa > N$ *and* $a + s > \frac{N-1}{2}$, *then*

$$\lim_{n\to\infty} E_n^s f(x) = f(x),$$

uniformly on S^N.

(ii) *If* $pa = N$ *and* $a + s > \frac{N-1}{2}$, *and in addition a function* f *above is continuous in the domain* $\Sigma \subset S^N$, *then*

$$\lim_{n\to\infty} E_n^s f(x) = f(x),$$

uniformly on any compact $K \subset \Sigma$.

(iii) *If* $f(x)$ *is vanishing in some domain* $\Sigma \subset S^N$ *and* $a + s > max\{\frac{N}{p} - 1, \frac{N-1}{2}\}$, *then*

$$\lim_{n\to\infty} E_n^s f(x) = 0,$$

uniformly on any compact $K \subset \Sigma$.

(iv) *If* $f(x)$ *is vanishing in some domain* $\Sigma \subset S^N$ *and also in its diametrically opposite* Σ^*. *and* $a + s > \frac{N-1}{2}$, *then*

$$\lim_{n\to\infty} E_n^s f(x) = 0,$$

uniformly on any compact $K \subset \Sigma$.

Theorem 2.1 provides sufficient conditions for the uniform convergence of $E_n^s f(x)$ functions from the class $H_p^a(S^N)$. A condition $ap > N$ is precise because in other case there exists an unbounded function in $H_p^a(S^N)$ whose Fourier-Laplace series trivially is divergent and cannot be uniformly summable on S^N. Item (ii) in the theorem shows that in critical case when $ap = N$ additional conditions gives positive answer for the uniform summability problem. This clarification methods developed in [1] and [6]. Inequality $a + s > max\{\frac{N}{p} - 1, \frac{N-1}{2}\}$, of the item (iii) of the theorem above makes corrections of the corresponding condition in theorem 2 of the paper [11]. Necessity of the conditions in theorem 2.1 is discussed in the the theorem below.

Theorem 2.2 *Let* $p \geq 1,\ a > 0.$

(i) *if* $s + \alpha = \frac{N}{p} - 1$, *then for any* $x_0 \in S^N$ *there is a function* $f \in H_p^a(S^N)$, *that is equal zero in some neighborhood of this point and satisfies the following inequality*

$$\overline{\lim_{n\to\infty}} E_n^s f(x_0) > 0.$$

(ii) *if* $s + \alpha = \frac{N-1}{2}$, *then for any* $x_0 \in S^N$ *there is a function* $f \in C^a(S^N)$, *that is equal zero in some neighborhood of this point as well as in some neighborhood of the diametrically opposite point and satisfies the following inequality*

$$\overline{\lim_{n\to\infty}} E_n^s f(x_0) > 0.$$

A condition $s + \alpha = \frac{N}{p} - 1$ in (i) of theorem 2.2 proves preciseness of the condition in item (iii) of the theorem 2.1. A condition $s + \alpha = \frac{N-1}{2}$ proves preciseness of the corresponding conditions in (i),(ii),(iii) and (iv)in the theorem 2.1. Moreover theorem 2.2 also corrects theorem 1 in [12].

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3. Preliminaries and proof of the theorem.

The proof of the main theorem based on number of supplementary statements. From the estimations (3), (4) and (5) we obtain

Lemma 1 *Let* $n > 1$, $\alpha > -1$ *and* $1 \le q \le \infty$ $(1/q + 1/p = 1)$. *Uniformly with the respect variable* x *we have*

$$\|\Theta^{\alpha}(x,y,n)\|_{L_q(S^N)} \le C(n^{N/p} + n^{(N-1)/2-\alpha}), \quad q \ne \frac{2N}{N+1+2\alpha} \tag{6}$$

$$\|\Theta^{\alpha}(x,y,n)\|_{L_q(S^N\cap\{\gamma(x,y)>\gamma_0>0\})} \le C(n^{N/p-1-\alpha} + n^{(N-1)/2-\alpha}), \quad q \ne \frac{2N}{N-1} \tag{7}$$

$$\|\Theta^{\alpha}(x,y,n)\|_{L_q(S^N\cap\{\pi-\gamma_0>\gamma(x,y)>\gamma_0>0\})} \le Cn, \tag{8}$$

where the norm in L_q *is taken with the respect to the variable* y.

Let τ is a positive number. Using equality (1) we can define powers of the sefadjoint operator $1 + \overline{-\Delta_s}$

$$(1 + \overline{-\Delta_s})^{\tau} f(x) = \sum_{k=0}^{\infty} (1+\lambda_k)^{\tau} \sum_{j=1}^{a_k} f_{k,j} Y_j^k(x). \tag{9}$$

Note that operator $(1 + \overline{-\Delta_s})^{\tau}$ implements isomorphic mapping between spaces $H_p^a(S^N)$ and $H_p^{a+2\tau}(S^N)$ by the modulo $C^{\infty}(S^N)$ [2].

Let $g(x) = (1 + \overline{-\Delta_s})^{\tau} f(x)$. Then we can obtain the new representation for $E_n^{\alpha} f(x)$

$$E_n^{\alpha} f(x) = \int_{S^N} \Theta_{\tau}^{\alpha}(x,y,n) g(y) d\sigma(y), \tag{10}$$

where $\Theta_{\tau}^{\alpha}(x,y,n)$ defined as follows

$$\begin{aligned}\Theta_{\tau}^{\alpha}(x,y,n) &= (1 + \overline{-\Delta_s})^{-\tau} \Theta^{\alpha}(x,y,n) \\ &= \sum_{k=0}^{n} \left(1 - \frac{\lambda_k}{\lambda_n}\right)^{\alpha} (1+\lambda_k)^{-\tau} \sum_{j=1}^{a_k} Y_j^{(k)}(x) Y_j^{(k)}(y).\end{aligned} \tag{11}$$

Using the Holder inequality we can estimate (10)

$$\begin{aligned}|E_n^{\alpha} f(x)| &\le \int_{S^N} |\Theta_{\tau}^{\alpha}(x,y,n)||g(y)| d\sigma(y) \\ &\le \left(\int_{S^N} |\Theta_{\tau}^{\alpha}(x,y,n)|^q d\sigma(y)\right)^{\frac{1}{q}} \|g\|_{L_p}, \quad \frac{1}{p} + \frac{1}{q} = 1.\end{aligned} \tag{12}$$

We choose $\tau = \frac{a-\varepsilon}{2}$, $\varepsilon > 0$. The estimation of $(\int_{S^N} |\Theta_{\tau}^{\alpha}(x,y,n)|^q d\sigma(y))^{\frac{1}{q}}$ follows from the Lemma 1. Then for the estimation of $\|g\|_{L_p}$ we apply theorem on isomorphism of the Nikolskii spaces [2]

$$\|g\|_{L_p} = \|(1 + \overline{-\Delta_s})^{\frac{a-\varepsilon}{2}} f\|_{L_p} \le C\|f\|_{H_p^{a-\varepsilon}} \le C\|f\|_{H_p^a}$$

This conclude the proof of the Theorem 1. Its statements follows from the statements of the Lemma 1 and estimation for $\|g\|_{L_p}$ above.

The proof of the theorem 2 is based on the analogue of the Lemma 3 in [5] for the Riesz means.

4. Discussions

The questions of the uniformly summability of the spectral expansions associated with the selfadjoint extensions of elliptic differential operators of arbitrary order in N-dimensional domain studied in [1]. These problems for the Fourier-Laplace series for the Cesaro means studied by Pulatov in [5]. The Fourier-Laplace series in the spaces of singular distributions studied in [3] and [9]. More general expansions for the singular distributions studied in [6], [7] and [8].

5. Conclusion

From the main statements of these paper and paper [5] we observe similar behavior of the Reisz means of the Fourier-Laplace series on the sphere and the Chezaro means similar in terms of uniform convergence.

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