Journal of Physics: Conference Series

PAPER • **OPEN ACCESS**

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IOP Conf. Series: Journal of Physics: Conf. Series **1489** (2020) 012006 doi:10.1088/1742-6596/1489/1/012006

A numerical solution for heat transfer past a stretching sheet with ohmic dissipation and suction or injection problem using Haar wavelet quasilinearization method

C G Nor Artisham[1] and M A Suazlan[2]

[1] Centre for Foundation Studies in Science, University of Malaya, 50603 Kuala Lumpur, Malaysia
[2] Science in Engineering Department, Faculty of Engineering, International Islamic University Malaysia, PO. Box 10, 50728 Kuala Lumpur, Malaysia

Abstract. This paper represents a numerical analysis for heat transfer of a Jeffrey fluid flow past a stretching sheet with ohmic dissipation and suction/injection. The partial differential equations are reduced into a set of convenient nonlinear ordinary differential equations with the boundary conditions. Haar wavelet quasilinearization method (HWQM) is used to solve ordinary differential equations. The effect of various related parameters on velocity and temperature profiles are computed and analyzed. Then, comparison is made between the numerical results of proposed method with existing numerical solutions found in the literature, and reasonable agreement is noted.

1. Introduction

The boundary layer flows induced by a stretching sheet has great importance in the aerodynamic extrusion of plastic sheets, crystal growing, continuous casting, cooling of metallic plate in a bath, glass fiber and paper production, the boundary layer along a liquid film in the condensation process and many others [1]. Such consideration in the presence of heat transfer has central role in the polymer industry. An exact analytic solution for the two dimensional boundary layer flow of viscous fluid over a linearly stretching surface was firstly presented by Crane [2]. Later, this problem has been extensively examined through various aspects such as suction/blowing, stretching velocities, magnetohydrodynamics, heat/ mass transfer and so on. Further, the addition of heat generation/absorption term in energy expression is very important in the cases involving underground disposal of radioactive waste material, storage of food stuffs, heat removal from nuclear fuel fragments and packed bed reactors. Some of the studies on such effects can be seen in Ref. [1, 3-7].

Related to the presence of suction/injection, Vajravelu [8] has applied variable size of finite difference method for solving the convection flow and heat transfer of a viscous fluid near an infinite, porous and vertical stretching surface. Muthucumaraswamy [9] studied the effects of suction on heat and mass transfer along a moving vertical surface in the presence of chemical reaction. El-Arabawy [10] investigated the effects of suction/injection and chemical reaction on mass transfer over a stretching surface. Elbashbeshy and Bazid [11] have analyzed the effect of internal heat generation and suction or injection on the heat transfer in a porous medium over a stretching surface. Sultana et al. [12] discussed the effects of internal heat generation, radiation and suction or injection on the heat transfer in a porous medium over a stretching surface. Rajeswari et al. [13] studied the effect of chemical reaction, heat and mass transfer on nonlinear MHD boundary layer flow through vertical porous surface with heat source in the presence of suction. Elbashbeshy et al. [14] used Runge-Kutta technique to study the effects of suction/injection and variable chemical reaction on mass transfer characteristics over unsteady stretching surface embedded in porous medium. Kumari and Nath [15] analyzed the significant results for heat transfer on a thin vertical cylinder in the presence of suction/injection. Siri et al. [16] studied the boundary layer flow of Maxwell fluid past a stretching surface in the presence of suction/injection using two different approaches, HWQM and RK Gill method. Some research were purposed, in Ref. [17 - 21] to examine the impact of suction/injection during the flow of a fluid in the case of different geometries. Later, many researchers [22-27] investigated the influences of ohmic and viscous dissipation on nanofluid flow under diverse conditions.

Based on our literature study, no research has been made so far to study suction/injection on a Jeffrey fluid flow past a steady stretching sheet by using our proposed method. The non-dimensionalized partial differential equations (PDEs) [28] were transformed into the ordinary differential equations (ODEs) and solved numerically by employing the HWQM. Chen and Hsiao [29] led the work using Haar wavelet to solve system analysis. They solved the lumped and the distributed-parameters systems using Haar operational matrix derived from an integration of Haar function. Later, numerous numerical method studies based on the Haar wavelet was conducted when solving differential and integral equations, partial differential equation, fractional calculus, hyperbolic heat conduction problem and finding numerical inversion of Laplace transform [30]. Recently, HWQM has been adopted to solve fractional non-linear equation, Burgers equation, and Cattaneo Christov heat flux problem [30]. For solving the nonlinear and PDEs, Bellman and Kalaba [31] proposed the quasilinearization method which is based on the Newton Raphson method.

2. Mathematical formulation

A steady two-dimensional boundary layer flow of an incompressible Jeffrey fluid flow over an impermeable stretched sheet with the presence of suction/injection is considered. Noted that the flow is generated by stretching the boundary sheet from a slit by commanding two equal and opposite forces. The flow region is exposed under uniform transverse magnetic fields $\vec{B} = (0, B_0, 0)$ and uniform electric field $\vec{E} = (0, 0, -E_0)$, for which satisfies the Maxwell's equation, such that $\nabla.\vec{B} = 0$ and $\nabla.\vec{E} = 0$. If the magnetic field is not strong enough, the electric and magnetic field follow the Ohm's law, $\vec{J} = \sigma(\vec{E} + \vec{q} \times \vec{B})$, where $\vec{J}$ is the Joule current, σ is the magnetic permeability and $\vec{q}$ is the fluid velocity. Under the above stated assumptions, the governing boundary layer equations for momentum and energy are [28]

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \,, \qquad \textbf{Error! Bookmark not defined.}(1)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{V}{1+\lambda_1}\left[\frac{\partial^2 u}{\partial y^2} + \lambda_2\left(u\frac{\partial^3 u}{\partial x \partial y} - \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y} + v\frac{\partial^3 u}{\partial y^3}\right)\right] + \frac{\sigma E_0 B_0}{\rho} - \frac{\sigma B_0^2}{\rho}u \,, \qquad \textbf{Error! Bookmark not defined.}(2)$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2} + \frac{\mu}{\rho c_p}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\sigma}{\rho c_p}\,(uB_0 - E_0)^2 \,. \qquad (3)$$

The velocity components along the x and y directions are denoted as u and v, respectively. V is the kinematic viscosity, λ_1 is the fluid relaxation time, λ_2 is the thermal relaxation time, T is the fluid temperature, k is the thermal conductivity, ρ is fluid density, μ is the dynamic viscosity and c_p is the specific heat at constant pressure.

The boundary conditions in the present problem are

$$u = ax, \quad v = v_0, \quad T = T_w \quad \text{at} \quad y = 0,$$
$$u \to 0, \quad \frac{\partial u}{\partial y} \to 0, \quad T \to T_\infty \quad \text{as} \quad y \to \infty$$

Error! Bookmark not defined.(4)

In the Eqn. (4), v_0 represents the velocity of suction/injection at the wall, T_w is the temperature at the wall, and T_∞ is the ambient fluid temperature.

By using these similarity transformation,

$$\eta = \sqrt{\frac{a}{V}}y, \quad \psi = \sqrt{aV}xf(\eta), \quad \theta = \left(\frac{T - T_\infty}{T_w - T_\infty}\right)$$

Error! Bookmark not defined.(1)

in which ψ is the stream function defined in the $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$. η is the similarity variable, f and θ are the dimensionless stream function and dimensionless temperature, respectively, Eqns. (2) and (3) can be reduced to a system ODEs as follows:

$$f'''(\eta) - (1+\lambda)\left(f'^2(\eta) - f(\eta)f''(\eta) - Ha^2\big(E_1 - f'(\eta)\big)\right) + \beta\left(f''^2(\eta) - f(\eta)f^{iv}(\eta)\right) = 0,$$

Error! Bookmark not defined.(2)

$$\theta''(\eta) + Pr\big(f(\eta)\theta'(\eta) - 2f'(\eta)\theta(\eta)\big) + PrE_c f''^2(\eta)PrE_c Ha^2(f'^2(\eta) - 2E_1 f'(\eta) + E_1^2) = 0\,,$$

Error! Bookmark not defined.(3)

and the transformed boundary conditions (4) are

$$\eta = 0: \; f = s, \quad f' = 1, \quad \theta = 1,$$
$$\eta \to \infty: \; f' \to 0, \quad f'' \to 0, \quad \theta \to 0.$$

Error! Bookmark not defined.(4)

The Prandtl number is given as $Pr = \frac{\mu c_p}{k}$, Eskert number, $Ec = \frac{a^2 l^2}{Ac_p}$, Hartmann number, $Ha = \sqrt{\frac{\sigma}{\rho a}}B_0$ and the local electrical parameter is given as $E_1 = \frac{E_0}{B_0 ax}$. The suction parameter is $s = -\frac{v_0}{\sqrt{cv}}$, where c is a positive constant, $s > 0$ (suction), $s < 0$ (injection) and $s = 0$ (impermeable surface).

3. Numerical solutions

This section presents the Haar wavelet quasilinearization method (HWQM) for solving nonlinear ODEs (6) – (8). Wavelet methods became a requisite mathematical tool in many investigations and have numerous applications including finding an approximate solution of differential equations. Among prominent wavelet basis used in numerical methods are Chebyshev-, Legendre-, Haar-, and Meyer-wavelets. However, the Haar wavelet has caught special attention for the reason that it is the simplest orthogonal wavelet. Other beneficial features using Haar wavelet basis are discussed in Refs. [32 - 34].

3.1 Haar wavelets

The Haar scaling function for $\eta \in [0, X]$ is defined as

$$h_0(\eta) = \begin{cases} 1 \ , & 0 \leq \eta < X, \\ 0 \ , & \text{elsewhere in } [0, X), \end{cases}$$

Error! Bookmark not defined.(9)

and corresponding Haar wavelet function

$$h_i(\eta) = \begin{cases} 1 & \dfrac{kX}{2^\alpha} \leq \eta < X \\ -1 & \dfrac{(k+1/2)X}{2^\alpha} \leq \eta < \dfrac{(k+1)X}{2^\alpha} \\ 0 & \text{elsewhere in } [0,X), \end{cases}$$

Error! Bookmark not defined.(10)

where $i = 1, 2, \ldots, m-1$ is the series index number and the resolution $m = 2^J$ is a positive integer and J is the maximum level of resolution. In Eqn. (10), α and k represent the integer decomposition of the index *i*, i.e. $i = 2^\alpha + k$ in which $\alpha = 0, 1, \ldots, J-1$ and $k = 0, 1, \ldots, 2^\alpha - 1$.

In the Haar wavelet method, the following integrals are used

$$p_{i,j}(\eta) = \int_0^{\eta} h_i(\eta')\, d\eta',$$

Error! Bookmark not defined.(11)

$$p_{i,j+1}(\eta) = \int_0^{\eta} h_{i,j}(\eta')\, d\eta' \text{ where } j = 1,2,3 \ldots.$$

Error! Bookmark not defined.(12)

The generalized Haar wavelet and its integration are derived and modified, hence the expansion of Haar series could be between zero to a number greater than one. It is a great help due to the boundary value problem always dealing with sufficiently a large interval $[0, X)$. Any square integrable function $f \in L^2\ [0, X]$ can be decomposed into a linear combination of Haar basis and can be written as

$$f(\eta) = \sum_{i=0}^{\infty} c_i h_i(\eta),$$

Error! Bookmark not defined.(13)

where the Haar coefficients, c_i can be obtained from

$$c_i = \frac{2^\alpha}{X} \int_0^X f(\eta) h_i(\eta)\, d\eta.$$

Error! Bookmark not defined.(14)

The above series terminates at finite terms if $f(\eta)$ may be approximated as piecewise constant during each subinterval.

3.2 Quasilinearization method

The nonlinear boundary value problems (BVPs) is linearized in the quasilinearization method. Hence, it forms a series of equations, which is typically convergence in nature [31, 35, 36].

We consider a nonlinear boundary value problems of second order differential equation,

$$\theta''(\eta) = f(\theta'(\eta), \theta(\eta), \eta). \quad \textbf{Error! Bookmark not defined.}(15)$$

subject to boundary conditions,

$$\theta'(a) = \kappa, \quad \text{and} \quad \textbf{Error! Bookmark not defined.}(16)$$

$$\theta(b) = \tau, \quad a \leq \eta \leq b, \quad \textbf{Error! Bookmark not defined.}(17)$$

where f is a nonlinear differentiable function. Let $\theta_0(\eta)$ be an initial approximation of the function $\theta(\eta)$. The Taylor series expansion of f around $\theta_0(\eta)$ is

$$f(\theta'(\eta), \theta(\eta), \eta) = f(\theta_0'(\eta), \theta_0(\eta), \eta) + \left(\theta_{(\eta)} - \theta_0(\eta)\right) f_{\theta_0}(\theta_0'(\eta), \theta_0(\eta), \eta) + (\theta'(\eta) - \theta_0'(\eta)) f_{\theta_0'}(\theta_0'(\eta), \theta_0(\eta), \eta) \quad \textbf{Error! Bookmark not defined.}(18)$$

where the function $f_\theta(\theta'(\eta), \theta(\eta), \eta) = \partial f/\partial\theta$ and $f_{\theta'}(\theta'(\eta), \theta(\eta), \eta) = \partial f/\partial\theta'$. Ignoring the second order and higher order terms in Equation (18). Then Eqn. (18) in Eqn. (15) yields

$$\theta''(\eta) = f(\theta_0'(\eta), \theta_0(\eta), \eta) + \left(\theta_{(\eta)} - \theta_0(\eta)\right) f_{\theta_0}(\theta_0'(\eta), \theta_0(\eta), \eta) + (\theta'(\eta) - \theta_0'(\eta)) f_{\theta_0'}(\theta_0'(\eta), \theta_0(\eta), \eta). \quad \textbf{Error! Bookmark not defined.}(19)$$

By solving Eqn. (19) for $\theta(\eta)$ and call it $\theta_1(\eta)$. Using $\theta_1(\eta)$ and again expanding Eqn. (15) about $\theta_1(\eta)$ yields

$$\theta''(\eta) = f(\theta_1'(\eta), \theta_1(\eta), \eta) + \left(\theta_{(\eta)} - \theta_1(\eta)\right) f_{\theta_1}(\theta_1'(\eta), \theta_1(\eta), \eta) + \left(\theta_\eta'(\eta) - \theta_1'(\eta)\right) f_{\theta_1'}(\theta_1'(\eta), \theta_1(\eta), \eta). \quad \textbf{Error! Bookmark not defined.}(50)$$

And the third approximation for $\theta(\eta)$, and call it $\theta_2(\eta)$. Assuming that the problem converges and in general, the recurrence relation can be written as

$$\theta_{r+1}''(\eta) = f(\theta_r'(\eta), \theta_r(\eta), \eta) + (\theta_{r+1}(\eta) - \theta_r(\eta)) f_{\theta_r}(\theta_r'(\eta), \theta_r(\eta), \eta) + (\theta_{r+1}'(\eta) - \theta_r'(\eta)) f_{\theta_r'}(\theta_r'(\eta), \theta_r(\eta), \eta), \quad \textbf{Error! Bookmark not defined.}(21)$$

with $r = 0, 1, 2, ...$ is number of iteration with

$$\theta'_{r+1}(a) = \kappa, \quad \text{and}$$ **Error! Bookmark not defined.**(22)

$$\theta_{r+1}(b) = \tau.$$ **Error! Bookmark not defined.**(23)

Eqn. (21) is all the time a linear and can be computed recursively using the Haar series [37]. The details of convergence for this method can be referred to in literature [35, 36].

Post quasilinearization of the Eqns. (6) and (7), yields

$$\alpha_{1,r} f_{r+1}^{(iv)} + \alpha_{2,r} f_{r+1}''' + \alpha_{3,r} f_{r+1}'' + \alpha_{4,r} f_{r+1}' + \alpha_{5,r} f_{r+1} = R_1 ,$$ **Error! Bookmark not defined.**(24)

$$\beta_{1,r} f_{r+1}'' + \beta_{2,r} f_{r+1}' + \beta_{3,r} f_{r+1} + \beta_{4,r} \theta_{r+1}'' + \beta_{5,r} \theta_{r+1}' + \beta_{6,r} \theta_{r+1} = R_2 ,$$ **Error! Bookmark not defined.**(25)

where

$$\alpha_{1,r} = \beta f_r^2,$$
$$\alpha_{2,r} = -f_r,$$
$$\alpha_{3,r} = -f_r(f_r + \lambda f_r + 2\beta f_r''),$$
$$\alpha_{4,r} = f_r(2f_r' + Ha^2 + 2\lambda f' + \lambda Ha^2),$$
$$\alpha_{5,r} = f_r''' - (1+\lambda) f_r'^2 + (1+\lambda) Ha^2 E_1 - Ha^2 f_r',$$
$$\beta_{1,r} = 2\, Pr\, E\, c f_r'',$$
$$\beta_{2,r} = 2\, Pr\big(-\theta_r + EcHa^2(f_r' - E_1)\big),$$
$$\beta_{3,r} = Pr\, \theta_r',$$
$$\beta_{4,r} = 1,$$
$$\beta_{5,r} = Pr\, f_r,$$
$$\beta_{6,r} = -2\, Pr\, f_r',$$
$$R_1 = 2Ha^2 E_1 f_r(1+\lambda) + f_r(f_r''' - (1+\lambda) Ha^2 f_r'),$$
$$R_2 = Pr\, E\, c\left(f_r''^2 + Ha^2\left(f_r'^2 - E_1^2\right)\right) + Pr(\theta_r' f_r - 2\theta_r f_r').$$

The boundary conditions are at

$$\eta = 0: \quad f_{r+1} = s, \quad f_{r+1}' = 1, \quad \theta_{r+1} = 1,$$ **Error! Bookmark not defined.** (26)

$$\eta \to \infty: \quad f_{r+1}' \to 0, \quad f_{r+1}'' \to 0, \quad \theta_{r+1} \to 0.$$ **Error! Bookmark**

IOP Conf. Series: Journal of Physics: Conf. Series **1489** (2020) 012006 doi:10.1088/1742-6596/1489/1/012006

not defined.

Haar wavelet method is applied to Eqns. (24) and (25), we get

$$f_{r+1}^{(iv)}(\eta) = \sum_{i=0}^{m-1} a_i h_i(\eta), \quad \text{and}$$ **Error! Bookmark not defined.**(27)

$$\theta_{r+1}''(\eta) = \sum_{i=0}^{m-1} b_i h_i(\eta).$$ **Error! Bookmark not defined.**(28)

The lower order derivatives are obtained as follows by integrating Eqns. (27) and (28).

$$f_{r+1}'''(\eta) = \sum_{i=0}^{m-1} a_i \left(\frac{2}{N^2} p_{i.3}(N) - \frac{2}{N} p_{i,2}(N) + p_{i,1}(\eta) \right) + \frac{2}{N^2},$$ **Error! Bookmark not defined.**(29)

$$f_{r+1}''(\eta) = \sum_{i=0}^{m-1} a_i \left(p_{i.2}(N) - \left(\frac{2\eta}{N} - 1 \right) p_{i,2}(N) + \frac{2}{N} \left(\frac{\eta}{N} - 1 \right) p_{i,3}(N) \right) + \frac{2\eta}{N^2} + \frac{2}{N} \tag{30}$$

$$f_{r+1}'(\eta) = \sum_{i=0}^{m-1} a_i \left(\begin{array}{c} p_{i.3}(\eta) + \frac{\eta^2}{N^2} \left(p_{i,3}(N) - N p_{i,2}(N) \right) \\ + \eta \left(-p_{i,2}(N) - \frac{2}{N} p_{i,3}(N) + 2 p_{i,2}(N) \right) \end{array} \right) + \frac{\eta^2}{N^2} - \frac{2\eta}{N} + 1 \tag{31}$$

$$f_{r+1}(\eta) = \sum_{i=0}^{m-1} a_i \left(\begin{array}{c} p_{i.4}(\eta) - \frac{\eta^3}{3N^2} \left(-p_{i,3}(N) + N p_{i,2}(N) \right) \\ + \frac{\eta^2}{2} \left(-p_{i,2}(N) - \frac{2}{N} p_{i,3}(N) + 2 p_{i,2}(N) \right) \end{array} \right) + \frac{\eta^3}{3N^2} - \frac{\eta^2}{N} + \eta + s \tag{32}$$

$$\theta_{r+1}'(\eta) = \sum_{i=0}^{m-1} b_i \left(p_{i,1}(\eta) - \frac{1}{N} p_{i,2}(N) \right) - \frac{1}{N}\ , \tag{33}$$

$$\theta_{r+1}(\eta) = \sum_{i=0}^{m-1} b_i \left(p_{i,2}(\eta) - \frac{\eta}{N} p_{i,2}(N) \right) - \frac{\eta}{N} + 1, \tag{34}$$

where N is sufficiently large number. Substitute Eqns. (27) – (34) into Eqns. (24) and (25). By applying discretization on the Eqns. (24) and (25) and use the collocations points, $\eta_c = \frac{(c+0.5)N}{m}$, $c = 0,1,\cdots,m-1$, we obtain the following systems,

$$\sum_{i=0}^{m-1} a_i K_1 = L_1,$$ **Error! Bookmark not defined.**(35)

$$\sum_{i=0}^{m-1} a_i K_2 + \sum_{i=0}^{m-1} b_i K_3 = L_2, \quad \text{Error! Bookmark not defined.}(36)$$

where

$$K_1 = \beta f_r^2 h_i(\eta) - f_r\left(p_{i,1}(\eta) + \frac{2}{N^2}p_{i,3}(N) - \frac{2}{N}p_{i,2}(N)\right) - f_r\left(f_r + \lambda f_r + 2\beta f_r''\right) \times \left(p_{i,2}(\eta) + \frac{2\eta}{N^2}p_{i,3}(N) - \frac{2\eta}{N}p_{i,2}(N) - p_{i,2}(N) - \frac{2}{N}p_{i,3}(N) + 2p_{i,2}(N)\right) + f_r\left(2f'_r + Ha^2 + 2\lambda f'_r + \lambda Ha^2\right)\left(p_{i,3}(\eta) + \left(\frac{\eta}{N}\right)^2 p_{i,3}(N) - \frac{\eta^2}{N}p_{i,2}(N) - \eta p_{i,2}(N) - \frac{2\eta}{N}p_{i,3}(N) + 2\eta p_{i,2}(\eta)\right)\left(f_r''' - f_r'^2 + Ha^2E_1 - Ha^2 f_r' - \lambda f_r'^2 + \lambda Ha^2E_1 - \lambda Ha^2 f_r' + \beta f_r'''^2\right)\left(p_{i,4}(\eta) + \frac{\eta^3}{3N^2}p_{i,3}(N) - \frac{\eta^3}{3N}p_{i,2}(N) - \frac{\eta^2}{2}p_{i,2}(N) - \frac{\eta^2}{N}p_{i,3}(N) + \eta^2 p_{i,2}(N)\right), \quad \text{Error! Bookmark not defined.}$$

$$K_2 = 2PrEcf_r''\left(p_{i,2}(\eta) + \frac{2\eta}{N^2}p_{i,3}(N) - \frac{2\eta}{N}p_{i,2}(N) - p_{i,2}(N) - \frac{2}{N}p_{i,3}(N) + 2p_{i,2}(N)\right) - 2Pr(\theta_r - EcHa^2f_r' + EcE_1Ha^2)\left(p_{i,3}(\eta) + \left(\frac{\eta}{N}\right)^2 p_{i,3}(N) - \frac{\eta^2}{N}p_{i,2}(N) - \eta p_{i,2}(N) - \frac{2\eta}{N}p_{i,3}(N) + 2\eta p_{i,2}(N)\right) + Pr\theta'_r\left(p_{i,4}(N) + \frac{\eta^3}{3N^2}p_{i,3}(N) - \frac{\eta^3}{3N}p_{i,2}(N) - \frac{\eta^2}{2}p_{i,2}(N) - \frac{\eta^2}{N}p_{i,3}(N) + \eta^2 p_{i,2}(N)\right),$$

$$K_3 = h_i(\eta) - 2Prf_r'\left(p_{i,2}(\eta) - \frac{\eta}{N}p_{i,2}(N)\right) + Prf_r\left(p_{i,1}(\eta) - \frac{1}{N}p_{i,2}(N)\right),$$

$$L_1 = 2Ha^2E_1 f_r(1+\lambda) + f_r\left(f_r^{'''} + \frac{2}{N^2}\right) - Ha^2 f_r f_r'(1+\lambda) - f_r(2f_r' + Ha^2 + 2\lambda f_r' + \lambda Ha^2)\left(\frac{\eta^2}{N^2} - \frac{2\eta}{N} + 1\right) - \left(f_r^{'''} - f_r^{'2} + Ha^2E_1 - Ha^2 f_r^{'} - \lambda f_r^{'2} + \lambda Ha^2 E_1 - \lambda Ha^2 f_r^{'} + \beta \boldsymbol{f_r''}^2\right)\left(\frac{\eta^3}{3N^2} - \frac{\eta^2}{N} + \eta + s\right) + f_r^2\left(1 + \lambda + \frac{2\beta}{f_r}\boldsymbol{f_r''}\right)\left(\frac{2\eta}{N^2} - \frac{2}{N}\right),$$

$$L_2 = PrEc\left(\boldsymbol{f_r''}^2 + Ha^2 f_r'^2 - E_1^2 Ha^2\right) + Pr(\theta_r' f_r - 2\theta_r f_r') - Pr\theta_r'\left(\frac{\eta^3}{3N^2} - \frac{\eta^2}{N} + \eta + s\right) - 2Pr\left(\frac{1}{2}\theta' - \theta_r + EcHa^2 f_r' - EcE_1Ha^2\right)\left(\frac{\eta^2}{N^2} - \frac{2\eta}{N} + 1\right) - \frac{4}{N}PrEc\boldsymbol{f_r''}\left(\frac{\eta}{N} - 1\right) + 2Prf_r'\left(-\frac{\eta}{N} + 1\right) + \frac{Pr}{L}f_r\ .$$

To obtain Haar coefficients, a_i and b_i, Eqns. (35) and (36) can be solved simultaneously.

4. Results and Discussions

The nonlinear ODEs (6) and (7) subjected to the boundary conditions (8) were numerically solved by means HWQM. All the computations are carried out by using MATLAB.

The numerical results for examining the surface friction coefficient and surface temperature gradient for different values of suction/injection parameter, s and elasticity number, β is shown as in Table 1. Referring to this table, the values of $f''(0)$ and $\theta'(0)$ are reduced as s increases. Table 2 shows the comparison of the present results by comparing with the published results for various values of parameter with the Newtonian fluid, β is equal to zero. A good agreement is obtained between previous results and present results for different values of Ha, Ec, E_1 and Pr.

Table 1 A table with variations of $-f''(0)$ and $-\theta'(0)$ when $Ha = \lambda = 0$, $Ec = E_1 = 1$ and $Pr = 3$.

s	$-f''(0)$			$-\theta'(0)$		
	$\beta = 1$	$\beta = 2$	$\beta = 3$	$\beta = 1$	$\beta = 2$	$\beta = 3$
-0.5	0.06232022	0.18374761	0.10506315	1.78345485	1.82252559	1.87981412
-0.3	0.02630196	0.36782901	0.11560706	1.94181132	2.08351599	2.07537062
0.0	0.70909591	0.58066866	0.50469544	2.13955960	2.29268996	2.37501346
0.3	0.74462245	0.59878632	0.51844900	2.58130462	2.76592253	2.85853625
0.6	0.77445969	0.61380622	0.52987424	3.09771174	3.30931226	3.41034400
1.0	0.80652589	0.62966799	0.54178347	3.88532509	4.12576487	4.23544956

Table 2 Local Nusselt number $-\theta'(0)$ for various values of Ha, Ec, E_1 and Pr when $\beta = 0$.

Ha	Ec	E_1	Pr	Pal & Mondal [22]	Ahmad & Wahid [28]	HWQM (Present Method)
0	0	0	3	2.509715	2.5097	2.50972924
			5	3.316479	3.3164	3.31644910
	1	1	3	1.745111	1.7451	1.74508913
			5	2.219381	2.2193	2.21933272
1	1	1	3	2.227830	2.2277	2.22964763
			5	2.916217	2.9160	2.91291680
		0	3	0.459953	0.4600	0.46004010
			5	0.366367	0.3666	0.36651854

Figures 1 and 2 depict the velocity and temperature profiles for various values of β and Ha. When $\beta = 0$, the fluid becomes the Newtonian fluid because the elastic force disappears. It is clear that with the increase of β for a few values of Ha, the velocity distribution shows increasing behaviour. Additionally, the thickness of the momentum boundary layer decreases with an increase in Ha. Physically, due to the presence of electromagnetic force given by Ha, the flow velocity has been retarded at any point of boundary layer. Characteristics of β on the temperature distribution is displayed in Fig. 2. For $Ha = 1$ and 2, the temperature distribution increases for large values of β, as shown in Fig. 2(b) and Fig. 2(c). The increase parameter of β corresponds to larger relaxation time which provides resistance to the fluid motion, hence as a result more heat is produced. Therefore, temperature distribution increases. In contrary, Fig. 2(a) reveals that when $Ha = 0$, the temperature distribution decreases as β increases.

Figures 3(a) and 3(b) show the velocity and the temperature profiles with respect to the suction/injection parameter, s. The fluid velocity and temperature field are found to decrease with increasing value of s. Suction $(s > 0)$ causes to decrease the velocity of the fluid in the boundary layer region. This effect acts to decrease the wall shear stress. On the other hand, increase in suction causes progressive thinning of the boundary layer.

The influence of Pr on temperature profile is shown in Fig.4. It is observed that an increase in Pr leads to reduction in temperature and thermal boundary layer thickness. It should be noted that the thermal diffusivity is weaker for larger Pr due to the fact that the rate of diffusion decreases.

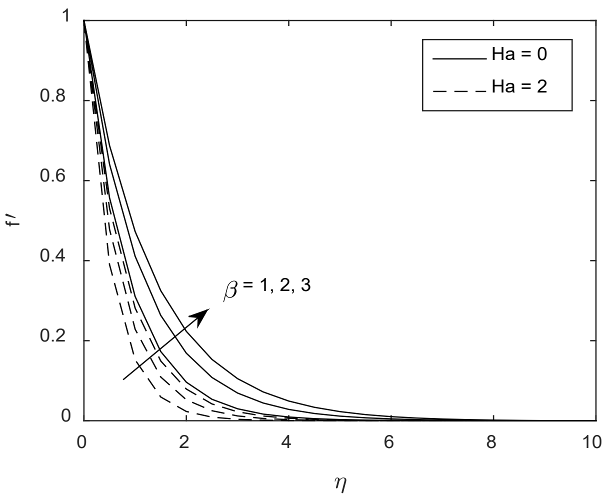

Fig. 1. The velocity profile for different values of β when $m = 128, N = 10, Ec = Pr = \lambda = s = 1$, $E_1 = 0$ and $Ha = 0, 2$.

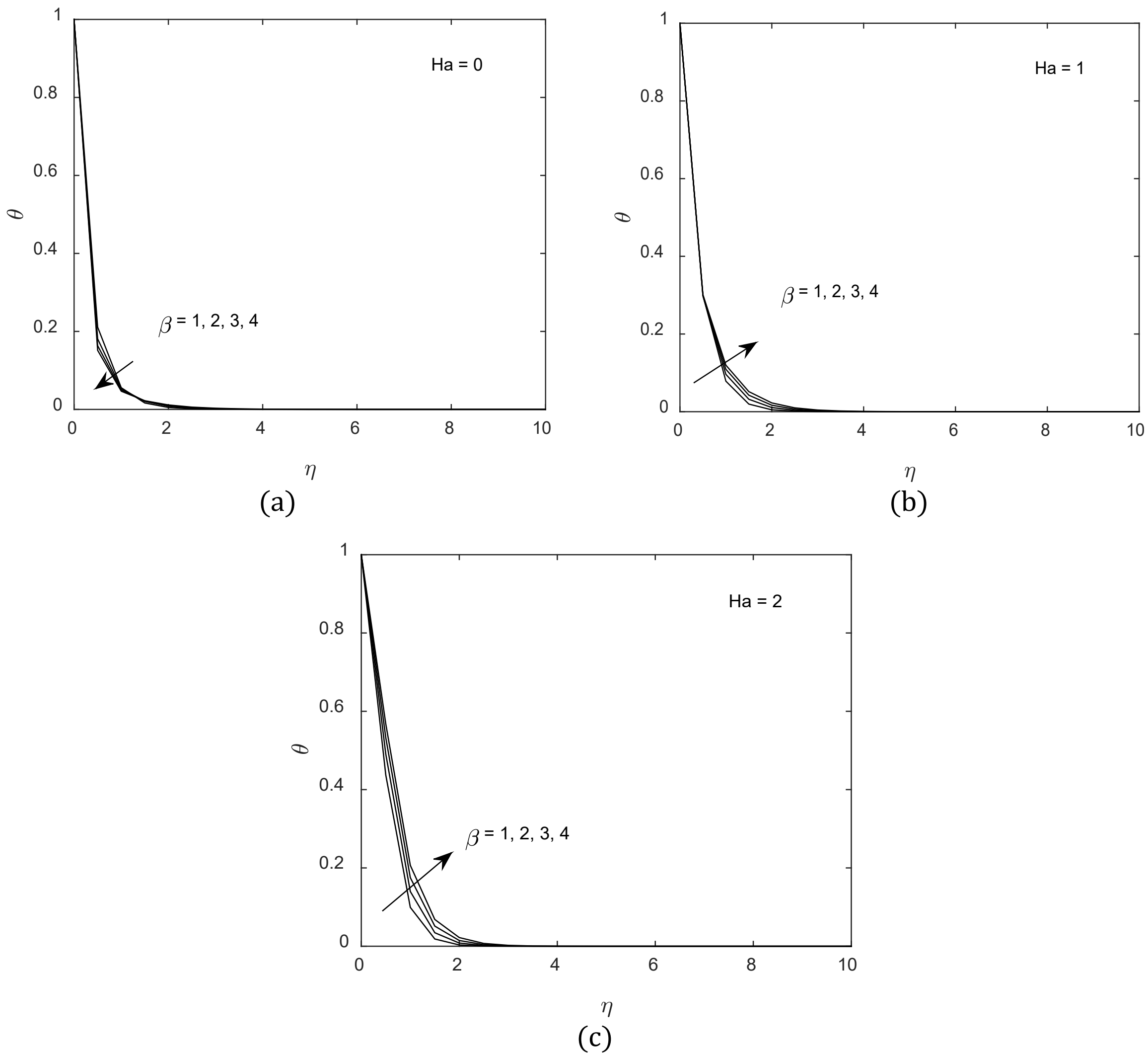

Fig. 2. The temperature profiles for (a) $Ha = 0$ (b) $Ha = 1$ (c) $Ha = 2$ when $m = 256, N = 10, Ec = \lambda = s = 1$ $\beta = 1, 2, 3, 4$, $Pr = 3$ and $E_1 = 0$.

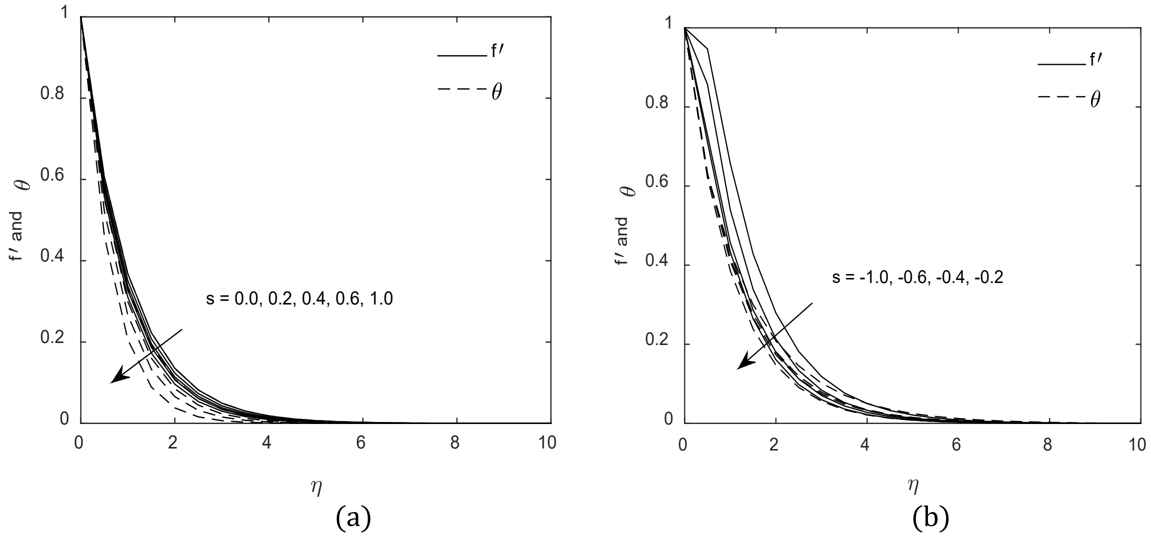

Fig. 3. The velocity and temperature profiles for (a) impermeable surface and suction (b) injection when $m = 256, N = 10, Ha = E_1 = 0, Ec = \beta = \lambda = Pr = 1$.

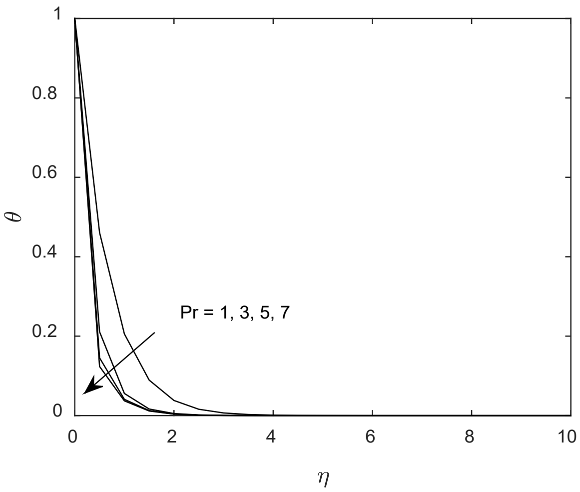

Fig. 4. The temperature profile for different values of Pr when $m = 256, N = 10, E_1 = Ha = 0, E_1 = \lambda = s = 1$.

5. Conclusions

The numerical analysis for heat transfer of a Jeffrey fluid flow past a stretching sheet with ohmic dissipation effect and suction/injection parameter is numerically studied by using HWQM. The results can be summarized as follows:

a) the velocity increases as β increases but it has opposite effects on temperature field, for $Ha = 0$,
b) temperature profile decreases as Pr increases and the temperature boundary layer becomes thinner,
c) the velocity and temperature distributions decrease on increasing the value of s,
d) $f''(0)$ decreases when suction/injection parameter increases.

Acknowledgments

The authors are appreciative of the International Islamic University Malaysia for financial support. This work was performed under IIUM Research Initiative Grant Scheme (RIGS) with Grant No. RIGS17-081–0656.

References

[1] Vajravelu K and Rollins D 1991**158** 1 pp 241-255
[2] Crane L J 1970 *J. Appl. Math. Phys*. **21** 4 pp 645-647
[3] Zheng L, Wang L and Zhang X 2011 *Commun. Nonlinear Sci*. **16** 2 pp 731-740
[4] Gireesha B J, Ramesh G K, Abel M S and Bagewadi C S 2011 *Int. J. Multiphas. Flow* **37** 8 pp 977-982
[5] Ramesh G K, Gireesha B J and Bagewadi C S 2012 *Int. J. Heat Mass Tran*. **55** 17-18 pp 4900-4907
[6] Shehzad S A, Alsaedi A, Hayat T and Alhuthali M S 2014 *PLoS ONE* **9** 1 0078240
[7] Cao L, Si X, Zheng L and Pang H 2015 *Bound. Value Probl*. **63** pp 1-18
[8] Vajravelu K 1994 *J. Math. Anal. Appl.* **188** 12 pp 1002-1011
[9] Muthucumaraswamy R 2002 *Forsch Ingenieurwes* **67** pp 129-132
[10] El-Arabawy H A M 2009 *J. Math. Stat*. **5** 3 pp 159-166
[11] Elbashbeshy E M Aand Bazid M A A 2004 *Appl. Math. Comput.* **158** 3 pp 799-807
[12] Sultana T, Saha S, Rahman M M and Saha G 2009 *J. Mech. Eng*. **40** 1 pp 22-28
[13] Rajeswari R, Jothiram B and Nelson V K 2009 *J. Appl. Math. Sci*. **3** 50 pp 2469-2480
[14] Elbashbeshy E M A, Emam T G and Abdel-wahed M S 2011 *Appl. Math. Sci*. **5** 12 pp557-571
[15] Kumari M and Nath G 2004 *Int. J. Heat Mass Transf.* **47** 5 pp 969–976
[16] Siri Z, Ghani N A C and Kasmani R M 2018 *Bound. Value Probl.* **2018** pp 126
[17] Ghani N A C and Siri Z 2018 *J. Phys. Conf. Ser.* **1139** 1 pp 1–8
[18] Pandey A K and Kumar M 2016 *Alexandria Eng. J*. **55** 4 pp 3115–3123
[19] Jalilpour B, Jafarmadar S, Ganji D D, Shotorban A B and Taghavifar H 2014 *J. Mol. Liq*. **195** pp 194–204
[20] Ganapathirao M and Ravindran R 2015 *Proc. of Int. Conf. Comput. Heat Mass Transf.* **127** pp 1102–1109
[21] Maity S, Singh S K and Kumar A V 2016 *J. Magn. Magn. Mater*. **419** pp 292–300
[22] Pal D and Mondal H 2017 *Commun. Nonlinear Sci. Numer. Simulat*. **15** pp 1197-1209
[23] Pal D and Mandal G 2015 *Powder Technol*. **279** pp 61–74
[24] Hayat T, Shafiq A and Alsaedi A 2016 *Alexandria Eng. J*. **55** 3 pp 2229–2240
[25] Upreti H, Pandey A K and Kumar M 2017 *Alexandria Eng. J*. **57** 3 pp 1839-1847
[26] Ganga B, Ansari S M Y, Ganesh N V and Hakeem A A 2015 *J. Nigerian Math. Soc*. **34** 2 pp 181–194
[27] Mishra A, Pandey A K and Kumar M *AIP Conf. Proc*. **9** 2 pp 99 – 115.
[28] Ahmad K and Wahid Z 2015 *Conf: Global Eng. Appl. Sci. Conf.*
[29] Chen C and Hsiao C 1997 *IEE Proc–Control Theory Appl*. **144** 1 pp 87
[30] Mt Aznam S, Ghani N A C and Chowdhury M S H 2019 *Results Phys.* **14** pp 102393
[31] Bellman R E and Kalaba R E 1965 American Elsevier Publishing Company
[32] Mt Aznam S 2012 M.Sc Thesis University of Malaya
[33] Ghani N A C 2012 M.Sc Thesis University of Malaya
[34] Ghani N A C 2018 Ph.D Thesis University of Malaya
[35] Mandelzweig V B and Tabakin F 2001 *Comput. Phys. Commun.* **141** pp 268–281
[36] Saeed U and Rehman M U 2013 *Appl. Math. Comput.* **220** pp 630-648
[37] Kaur H, Mittal R C and Mishra V 2014 **38** 21–22 pp 4958–4971