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OPTIMIZATION OF THE REGULARIZATION OF THE SOLUTION TO PLATE HEAT TRANSFER PROBLEMS

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ABSTRACT: Optimization of the regularization of the Fourier series in the case of a steady state heat transfer plate and heat transfer insulated plate problems are investigated and the regularization of the series solutions at a fixed point on the plates are studied at initial time and critical index.

ABSTRAK: Pengoptimuman aturan siri Fourier telah dikaji pada keadaan tetap masalah plat pindah haba dan plat penebat pindah haba. Penyelesaian bersiri secara aturan pada titik tetap plat telah dikaji pada masa mula dan indeks penting.

KEYWORDS: *optimization; regularization; heat transfer; initial time; critical index*

1. INTRODUCTION

Heat transfer problems confront researchers in many branches of science and engineering. Many researchers use Fourier transforms to solve the heat problem in different kinds of engineering applications such as in [1], where the problem of a hydromagnetic hot two-dimensional laminar jet supplying quiescent fluid of a lower temperature is studied using the Fourier series method. Therefore, it is shown that the process which is based on the Fourier series is very efficient and appropriate to solve boundary layer equations applied to plane jet flows with high accuracy. A temperature Fourier series solution in a hollow sphere subjected to periodic boundary conditions is studied in [2]. In this case, the material of the sphere is presumed to be homogeneous and isotropic with time-independent thermal properties [2]. The main purpose of [3] is to identify the flow types and exhibit augmented heat transfer in dependence on a magnetic induction gradient using Fast Fourier Transform analysis. Fourier transform blackbody spectroscopy to measure the radiation from blackbody sources operated at a series of various temperatures is studied in [4]. In [5], the author used a three-dimensional Fourier series solution to discuss the thermal characterization of electronic packages. Here, a Fourier series solution is used for computing local temperature distributions, heat fluxes and thermal resistances. Optimization of the regularized Fourier series solution for a fixed point plate vibration problem is studied at initial time and critical index [6].

The solutions of all above mentioned problems are based on the application of the Fourier series and transformations. Therefore, a regular summation method is required in

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the case of some singularities within the problem's input data. An example of this is when input data is expressed by singular distributions. In this paper, we will classify the singularity in terms of the Sobolev spaces and the Reisz method of summation will be considered as regularization of the Fourier series solutions to the problems.

2. PRELIMINARIES

Convergence or divergence of the Fourier series of an integrable function at a certain point depends only from the behaviour of the function in an arbitrary small neighbourhood of that point (localizations principles). When both expansions (Fourier series and integral) converge or diverge at the same time and same term, it is called equiconvergence. In the Liouville space $L_2^l(T^N)$, for any arbitrary self-adjoint elliptic operators, the sufficient condition for the localization of the Reisz means of order s of multiple Fourier series and integrals is [7]

$$l+s \geq \left\{\frac{N-1}{2}, \frac{N-1}{p}\right\}, \quad 1 \leq p \leq \infty \tag{1}$$

But for any elliptic operator, a sufficient localization condition is $l+s \geq \frac{N-1}{p}$ [7].

Moreover, the sufficient localization condition (1) can be weakened if summation of the multiple Fourier integrals occurs by the surface level of the elliptic operators from the class A_r,

$$l+s \geq \frac{N-1}{p} - r\left\{\frac{1}{p} - \frac{1}{2},\right\}, \quad 1 \leq p \leq \infty \tag{2}$$

In N-dimension, equiconvergence for both expansions (Fourier series and integral) is not valid for the rectangular partial sums. Equiconvergence in summation of the Fourier expansion of the linear continuous functional that is associated with an elliptic polynomial is discussed in [8]. Moreover, the general expanded expansions of the distributions are discussed in [9-19].

Let $\varepsilon(T^N)$ be the space of an infinitely differentiable function φ: $T^N \rightarrow$ C. The system of semi norm for any compact subset K of $T^N = [-\pi, \pi]^N$ is defined by,

$$P_{k,\gamma}(\phi) = \sup_{x \in k} \left| D^\gamma \phi(x) \right| \tag{3}$$

where γ is a multi-index denoted by $|\gamma| = (\gamma_1 + \gamma_2 + \ldots\ldots + \gamma_N)$ and $\gamma = (\gamma_1, \gamma_2, \ldots\ldots \gamma_N)$ is N dimensional vector with the non-negative integer components γ_j (j = 1, 2 …., N). For instance, $D^\gamma = D_1^{\gamma_1} D_2^{\gamma_2} \ldots\ldots\ldots D_N^{\gamma_N}$, where $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}, j = 1,2,\ldots\ldots N$.

For any functional $f \in \varepsilon'(T^N)$ (where $\varepsilon'(T^N)$ is the conjugate space of the locally convex topological space $\varepsilon(T^N)$), we can write

$$f = (2\pi)^{-N/2} \sum_{n \in Z^N} f_n e^{inx}, \tag{4}$$

where z^N is the set of all vectors with integer components, f_n is the Fourier coefficient, which is defined as the value of *f* on the test function on $(2\pi)^{-N/2}e^{-inx}$ and $x \in T^N$.

Consider the following elliptic polynomial:

$$A(n) = \left(\sum_{j=1}^{r+1} n_j^{\,2}\right)^{m+1} + \left(\sum_{j=r+2}^{N} n_j^{\,2}\right)^{m}\left(\sum_{j=1}^{N} n_j^{\,2}\right)$$

where $n = (n_1, n_2, ...n_N) \in Z^N$. m is a positive integer number and $r = 0, 1, 2, \ldots\ldots, N-1$. Polynomial $A(n)$ is a homogeneous of degree $2(m+1)$ and an elliptic ($A(n) > 0$). Note that, $A(n) \in A_r$ but $A(n) \notin A_{r+1}$ [20].

For any non-negative real number s, the Riesz means of order s of the Fourier series in equation (1) is defined as,

$$\sigma_\lambda^s f(x) = (2\pi)^{-\frac{N}{2}} \sum_{A(n)<\lambda} \left(1 - \frac{A(n)}{\lambda}\right)^s f_n \exp(inx). \tag{5}$$

Now, we extend a distribution *f* from N - dimensional torus T^N to the whole space R^N by zero. Then the Bochner-Riesz means of order *s* of the Fourier integral of *f* is,

$$R_\lambda^s f(x) = (2\pi)^{-\frac{N}{2}} \int_{A(\xi)<\lambda} \left(1 - \frac{A(\xi)}{\lambda}\right)^s \hat{f}(y) \exp(i\xi\, x) d\xi, \tag{6}$$

where, $\hat{f}(y) = < f, (2\pi)^{-N/2} \exp(-i\xi x) >$ is the Fourier transformation of the extended functional *f* and it acts on , $(2\pi)^{-N/2} \exp(-i\xi\, x)$ via x.

For any real number l, the Sobolev space of distributions $L_2^l(T^N)$ is

$$L_2^l(T^N) = \{ f \in \varepsilon' : \sum_{n \in Z^N} (1 + |n|^2)^l f_n^2 < \infty \} \tag{7}$$

Theorem 1

We have $\varepsilon(T^N) \subset \bigcup_{l=-\infty}^{\infty} L_2^{-l}$. The Dirac delta function $\delta \in L_2^{-l}$ (where $l > N/2$), and $s > \max\{\frac{(N-r-1)(1-1/2m)}{2} + \frac{r}{2}, \frac{N-1}{2}\} + l$ .Then for any $f \in L_2^{-l}(T^N)$

$$\sigma_\lambda^s f(x) = R_\lambda^s f(x) + O(1)\|f\|_{-l} \tag{8}$$

where $\|.\|_{-l}$ is a norm in $L_2^{-l}(T^N)$:

$$\|f\|_{-l} = (2\pi)^{-\frac{N}{2}} \sqrt{\sum_{n \in Z^N} (1 + |n|^2)^{-l} f_n^{\,2}} \text{ [8]} \tag{9}$$

In [21,22] mathematical models of thermo control processes in a rectangular plate are studied. In [21] it is considered that the temperature in a plate is controlled by heat exchange through one boundary while the other three are insulated. In these two papers, they study sufficient conditions for achieving the given projection by controlling the parameter on the boundary. Mathematical models of thermo control processes when control parameter is a vector function are studied in [23]. In [24] a control of the heat transfer was studied based on time/norm and it found sufficient and necessary conditions

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for such a control. Equiconvergence of the spectral expansions associated with the operators with singular coefficients was studied in [25] and for the Sturm Liouville operator in [26]. The definition of the operator $A(n)$ and application of the Poisson summation method is explained in [27].

In the next sections, for a steady-state heat transfer problem and insulated heat problems, we will verify this theorem numerically for different plates subjected to different boundary conditions. As mentioned, the Regular summation method will be used for excellent accuracy and convergence.

3. DESCRIPTION OF THE PROBLEM

Consider a thin square plate made of some thermally conductive material. Suppose the dimensions of the plate are $X \times Y$. The plate is heated in some way. For example, in a steady state heat transfer problem or heat transfer insulated (no heat escapes from this surface) plate problems. The plate is homogeneous, that is -its mass per unit volume is a constant.

Assume that,

$v(x, y, t)$ = temperature of the plate at position (x, y) and time t.

For a fixed t, $v(x, y, t)$ gives the temperature of the plate at position (x, y).

For the thin plate in which the temperature v is a function of time t and position (x, y) satisfy the two-dimensional heat equation

$$v_t = h(v_{xx} + v_{yy}), \qquad 0 < x < b, \quad 0 < y < c \tag{10}$$

where h is the heat conductivity coefficient and it is constant.

The solution of the heat equation (10) is subject to the boundary conditions and initial conditions.

The solution of the heat transfer problems in the two dimensional case has a form of a double Fourier series by eigenfunctions and these eigenfunctions depend on the boundary conditions. The coefficient of the Fourier series is found from the initial conditions. That is why convergence or divergence of the corresponding Fourier series depends on the smoothness or singularity of the initial conditions. If the initial conditions are given by very good functions then this series converges. In many engineering phenomena, the initial data may be not represented by good functions. Sometimes initial conditions may not even be functions. For example, initial conditions can be expressed as the Dirac delta function. In a one dimensional case, the Fourier series of the Dirac delta function diverges at a regular point. However, the arithmetic means of the Fourier series of the Dirac delta function converges. Thus, for the solutions of the corresponding heat problems some regularizations of the Fourier series solutions are required. Here, based on the singularity, we consider the Reisz method of summation as regularization of the Fourier series solutions of the heat problems. When we increase the order of the Reisz means, the solutions will converge but the numerical calculations will be increase. So, the regularized Fourier series solutions need optimization.

Optimization of the regularization of the solutions of the plate heat transfer problems is finding the solutions from the regularized Fourier series for minimum order of the Reisz means. The minimum order is $s > (N - 1)/2 + \quad l$. Here, s is the order and N is dimension.

After optimization, we will calculate the numerical solutions. Here, we will find the optimization of the regularization of the series solutions at a fixed point on the plates at the initial time and critical index. Finally, we will verify if the solution is convergent or not.

3.1 Problem 1: Steady-State Heat Transfer Problem

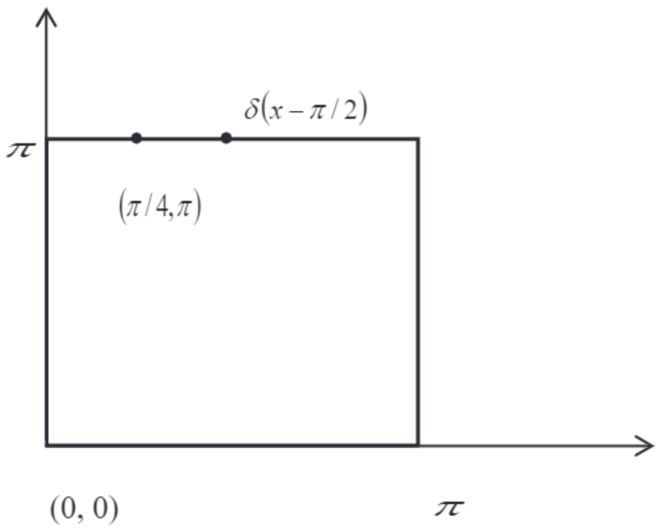

Fig. 1: A plate of dimensions $\pi \times \pi$ in the *xy*-plane.

Consider an aluminium plate of dimensions $\pi \times \pi$ in the *xy*-plane as shown on Fig. 1. The steady state equation subjected to the boundary conditions is

$$u_{xx}+u_{yy}=0, \qquad 0<x<\pi, \quad 0<y<\pi \tag{11}$$

$$u(0,y)=0, \tag{12}$$

$$u(x\ ,\ 0)=0, \tag{13}$$

$$u(\pi,y)=0, \tag{14}$$

$$u(x,\pi)=\delta\left(x-\frac{\pi}{2}\right), \tag{15}$$

where δ is a Dirac delta function.

Solution of the steady state problem is,

$$u(x,y)=\sum_{n=1}^{\infty}A_n Sin\,h(ny)\,Sin(nx), \tag{16}$$

where, A_n is a constant coefficient. After putting the boundary condition in equation (16) we found the value of A_n

$$A_n=\frac{Sin\frac{n\pi}{2}}{Sinh(n\pi)}\ , \tag{17}$$

Now, the solution of the steady state problem is,

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$$u(x,y)=\sum_{n=1}^{\infty}\frac{Sin\frac{n\pi}{2}}{Sinh(n\pi)}Sinh(ny)\,Sin(nx)\,, \tag{18}$$

The steady state solution at $y = \pi$ is

$$u(x,\pi)=\sum_{n=1}^{\infty} Sin\frac{n\pi}{2}Sin(nx)\,, \tag{19}$$

Therefore, to find the convergence of the series we have to choose a point. We know that in the point $\left(\frac{\pi}{2}\,,\pi\right)$, the value of the series is infinity. Now, take a point in $x = \pi$ axis except the Dirac delta function. Here we choose $x = \frac{\pi}{4}$ and $y = \pi$.

It should not be convergent. Here, regularization is required in order to solve this series in a stable manner. The Riesz means of order s (s is non-negative real number) of the Fourier series in equation (19) is defined as

$$E_{\lambda}^{s}u(x,\pi)=\sum_{|n|<\lambda}\left(1-\frac{|n|}{\lambda}\right)^{s} Sin\frac{n\pi}{2}\,Sin(nx) \tag{20}$$

3.1.1 Regularization of the Steady-State Heat Transfer Plate Problem at Initial Time

Here, we will test the regularization of the series in equation (20). We set order $s = 0$, 1 and time $t = 0$. The series solution in equation (20) for different values of λ and at the point $\left(\frac{\pi}{4}\,,\pi\right)$ is given below:

Table 1: Solution of the regularized series in equation (20) for different s ($s = 0$, $s = 1$) and time $t = 0$.

λ	$E_{\lambda}^{0}(\pi/4,\pi,0)$	$E_{\lambda}^{1}(\pi/4,\pi,0)$
350	-0.707106781	0.002020305
450	0.707106781	0.001571348
550	-0.707106781	0.001285649
650	0.707106781	0.001087857
750	-0.707106781	0.000942809
850	0.707106781	0.000831891
950	-0.707106781	0.000744323
1050	0.707106781	0.000673435
1150	-0.707106781	0.000614875
1250	0.707106781	0.000565685
1350	-0.707106781	0.000523783
1450	0.707106781	0.000487663

From Table 1, it is clear that for $s = 0$ the Fourier series diverges. After regularization, the Reisz means of order $s = 1$ of the Fourier series is convergent. The solution of the series of the Riesz means of order $s = 1$ is approximately very near to zero.

3.1.2 Regularization of the Steady-State Heat Transfer Plate Problem at Critical Index

From Table 1, it is clear that when s = 1, the series converges. We know that the series will converge when $s > \frac{N-1}{2} + 1$. So s = 1/2 is the critical point for the regularized series. Here, we choose a more complicated point, the critical point s = 0.5. To understand the difference between values below the critical point and values above the critical point, we take two more points: 0.5 – ε and 0.5 + ε (ε is very small number). Here, we choose ε = 0.1.Where all the other parameters are kept the same.

Table 2: Solution of the regularized series in equation (20) for different s (s = 0.4, s = 0.5, s = 0.6).

λ	$E_\lambda^{0.4}(\pi/4, \pi, 0)$	$E_\lambda^{0.5}(\pi/4, \pi, 0)$	$E_\lambda^{0.6}(\pi/4, \pi, 0)$
650	0.041758111	0.019793588	0.009259818
700	0.020462554	0.013759143	0.008979066
850	0.037451159	0.017249285	0.007826599
1300	-0.015876433	0.009997791	0.006101631
1450	0.030173547	0.013132134	0.005611037
1900	0.013602391	0.008230995	0.004822541
3700	-0.010381654	0.005860529	0.003197979
4250	0.019565055	0.076112881	0.002889619
9500	0.007096985	0.003635011	0.001795733

From Table 2, it is clear that below the critical point, the series diverges and after the critical point, the series is converges. But, at the critical point, the answer of the regularized series is so near to zero but still diverges.

3.2 Problem 2: Heat Transfer Problem of the Insulated Plate (when the Lower Plate is Insulated)

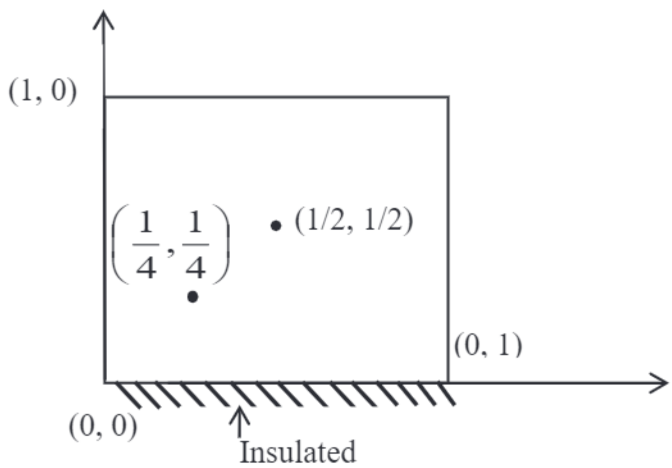

Fig. 2: An aluminium plate of dimensions 1 × 1 in the xy-plane.

Consider a square aluminium plate of dimensions 1 × 1 in the xy-plane as shown in Fig. 2. The plate is insulated in the x-axis. The heat transfer equation subjected to boundary conditions is

$$u_t = k(u_{xx} + u_{yy}), \quad 0 < x < 1, \quad 0 < y < 1 \tag{21}$$

$$u\left(0,y,t\right)=0, \tag{22}$$

$$u_y\left(x,0,t\right)=0, \tag{23}$$

$$u\left(1,y,t\right)=0, \tag{24}$$

$$u\left(x,1,t\right)=0, \tag{25}$$

$$u\left(x,y,0\right)=\delta\left(x-\frac{1}{4},y-\frac{1}{4}\right), \tag{26}$$

where δ is a Dirac delta function.

Solution of the steady state problem is,

$$u\left(x,y,t\right)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}U_{n_m}e^{-k(n^2+\left(\frac{2m+1}{2}\right)^2)\pi^2 t}Sin\left(n\pi x\right)Cos\left(\frac{2m+1}{2}\right)\pi y, \tag{27}$$

where, U_{n_m} is a constant coefficient. After putting the boundary condition in equation (27) we can find

$$U_{n_m}=Sin\frac{n\pi}{4}Cos\left(\frac{2m+1}{8}\right)\pi, \tag{28}$$

Now, the solution of the heat transfer problem is,

$$u\left(x,y,t\right)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}e^{-k(n^2+\left(\frac{2m+1}{2}\right)^2)\pi^2 t}Sin\frac{n\pi}{4}Cos\left(\frac{2m+1}{8}\right)\pi\, Sin\left(n\pi x\right)Cos\left(\frac{2m+1}{2}\right)\pi y \tag{29}$$

In this problem, we know that at the point $\left(\frac{1}{4},\frac{1}{4}\right)$, the value of the series is infinity. Now, choose a point in the plate except the Dirac delta function. Here, we take $x=\frac{1}{2}$ and $y=\frac{1}{2}$.

In this case, the input data has some singularities. It should not be convergent. Here, regularization is required in order to solve this series in a stable manner. The Riesz means of order s (s is non-negative real number) of the Fourier series in equation (29) is defined as

$$E_\lambda^s u\left(x,y,t\right)=\sum_{\sqrt{n^2+m^2}<\lambda}\left(1-\frac{\sqrt{n^2+m^2}}{\lambda}\right)^s e^{-k(n^2+\left(\frac{2m+1}{2}\right)^2)\pi^2 t}Sin\frac{n\pi}{4}Cos\left(\frac{2m+1}{8}\right)\pi\, Sin\left(n\pi x\right)Cos\left(\frac{2m+1}{2}\right)\pi y \tag{30}$$

3.2.1 Regularization of the Two Dimensional Heat Transfer Problem at Initial Time

Here, we will test the regularization of the series in equation (30). Therefore, we take order s = 0, 1, 2 and time t = 0. The series solution in equation (30) for different value of λ and at the point $\left(\frac{1}{2},\frac{1}{2}\right)$ is given below:

Table 3: Solution of the regularized series in equation (30) for different s (s = 0, s = 1, s = 2) and time t = 0.

λ	$E_\lambda^0(1/2,1/2,0)$	$E_\lambda^1(1/2,1/2,0)$	$E_\lambda^2(1/2,1/2,0)$
1700	-4.889995713	0.008050281	0.002943257
2500	-9.416538804	-0.003613664	0.001310513
5500	4.039739941	0.001821067	0.000226622
6500	11.13298168	-0.003669672	0.000153341
8500	-14.55226918	0.003217023	0.000077823
10500	17.22542063	-0.002024961	0.000047231
12500	-27.03824111	0.001093712	0.000040143
12700	30.23102475	-0.003398458	0.000039281
12900	9.304453421	0.001452938	0.000035381
13400	-15.70031947	-0.000344863	0.000030437

From Table 3, it is clear that for s = 0 the Fourier series diverges. Although after regularization the value of the Riesz means of order s = 1 of the Fourier series is approximately near to zero but the series is diverges. Finally, when the Reisz means of order s = 2, the series converges.

3.2.2 Regularization of the Square Plate Steady-State Heat Transfer Problem at Critical Index

From Table 3, it is clear that the series converges for s = 2. We know that the series converges when order $s > \frac{N-1}{2} + 1$. So, s = 3/2 is the critical point for the regularized series. Here, we choose the critical point s = 1.5. To understand the difference between below the critical point and above the critical point we take two more points 1.5 – ε and 1.5 + ε (ε is very small number). Here, we choose ε = 0.1.Where all the other parameters are kept the same.

Table 4: Solution of the regularized series in equation (30) for different s (s = 1.4, s = 1.5, s = 1.6).

λ	$E_\lambda^{1.4}(1/2,1/2,0)$	$E_\lambda^{1.5}(1/2,1/2,0)$	$E_\lambda^{1.6}(1/2,1/2,0)$
500	0.005020001	0.006763001	0.009379112
6500	0.000085401	0.000119112	0.000134201
7500	0.000150021	0.000130211	0.000118011
10500	-0.000038311	0.000006641	0.000066401
10520	0.000114353	0.000069814	0.000050122
10550	-0.000003256	0.000024155	0.000024811
10650	0.000117794	0.000069664	0.000020271
10800	-0.000057245	-0.000002461	0.000011784
18500	0.000004991	0.000007342	0.000009682
33000	-0.000000511	0.000009821	0.000000756

From Table 4, we can conclude that below the critical point, the series diverges and after the critical point, the series converges. But, at the critical point, the answer of the regularized series is so near to zero but it diverges.

3.3 Problem 3: Heat Transfer Problem of the Insulated Plate (when the Upper Plate is Insulated)

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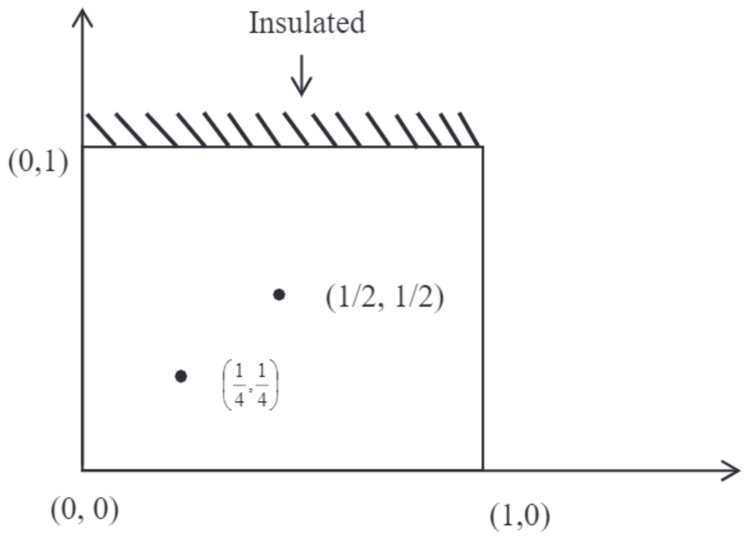

Fig. 3: A square plate of dimensions 1 × 1 in the *xy*-plane.

Consider a square aluminium plate of dimensions 1 × 1 in the *xy*-plane as shown on Fig. 3. The plate is insulated in the $x = 1$ axis. The heat transfer equation subjected to boundary conditions is

$$u_t = k(u_{xx} + u_{yy}), \qquad 0 < x < 1, \quad 0 < y < 1 \tag{31}$$

$$u\,(0, y, t) = 0, \tag{32}$$

$$u\,(x\,, 0, t) = 0, \tag{33}$$

$$u(1, y, t) = 0, \tag{34}$$

$$u_y\,(x, 1, t) = 0, \tag{35}$$

$$u(x, y, 0) = \delta\left(x - \frac{1}{4}, y - \frac{1}{4}\right), \tag{36}$$

where, δ is a Dirac delta function. Solution of the heat transfer equation of the insulated plate subjected to the boundary condition is,

$$u(x, y, t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} V_{n_m} e^{-k(n^2 + \left(\frac{2m+1}{2}\right)^2)\,\pi^2 t}\, Sin\,(n\pi x)\, Sin\left(\frac{2m+1}{2}\right)\pi y, \tag{37}$$

where, V_{n_m} is a constant coefficient. From the Dirac delta function and the boundary condition, we found

$$V_{n_m} = Sin\frac{n\pi}{4}\, Sin\left(\frac{2m+1}{8}\right)\pi, \tag{38}$$

Finally, the solution of the heat transfer problem is,

$$u(x, y, t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} e^{-k(n^2 + \left(\frac{2m+1}{2}\right)^2)\pi^2 t}\, Sin\frac{n\pi}{4}\, Sin\left(\frac{2m+1}{8}\right)\pi\, Sin\,(n\pi x)\, Sin\left(\frac{2m+1}{2}\right)\pi y \tag{39}$$

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In the point $\left(\frac{1}{4},\frac{1}{4}\right)$, the temperature of the plate is infinity. Here, we choose another point in the plate except the Dirac delta function. Therefore, we take $x=\frac{1}{2}$ and $y=\frac{1}{2}$. In the above series, the input data has some singularities. It should not be convergent. Here, regularization is required in order to solve this series in a stable manner. The Riesz means of order s (s is non-negative real number) of the Fourier series in equation (39) is defined as

$$E_{\lambda}^{s}u(x,y,t)=\sum_{\sqrt{n^2+m^2}<\lambda}\left(1-\frac{\sqrt{n^2+m^2}}{\lambda}\right)^{s}e^{-k\left(n^2+\left(\frac{2m+1}{2}\right)^2\right)\pi^2 t}\,Sin\frac{n\pi}{4}\,Sin\left(\frac{2m+1}{8}\right)\pi\;Sin\,(n\pi x)\;Sin\left(\frac{2m+1}{2}\right)\pi y \tag{40}$$

3.3.1 Regularization of the Two Dimensional Heat Transfer Problem at Initial Time

Now, we will test the regularization of the heat transfer plate. Here, we take order s = 0, 1, 2 and time t = 0. For different values of λ and a fixed point $\left(\frac{1}{2},\frac{1}{2}\right)$, the series solution (40) is given below:

Table 5: Solution of the regularized series in equation (40) for different s (s = 0, s = 1, s = 2) and time t = 0.

λ	$E_{\lambda}^{0}(1/2, 1/2, 0)$	$E_{\lambda}^{1}(1/2, 1/2, 0)$	$E_{\lambda}^{2}(1/2, 1/2, 0)$
500	2.75804036	0.01009106	0.01553744
1500	1.58279358	-0.00267634	0.00178935
1700	-2.56669864	0.00588032	0.00140549
2100	21.1892037	-0.00640545	0.00093207
2300	-12.4315789	0.00600931	0.00078491
2500	3.13509121	-0.00789911	0.00067197
3500	-19.2678209	0.00344795	0.00035346
5500	18.9915902	-0.00108365	0.00015099
8500	-24.9696107	0.00417739	0.00006941
10500	34.1948078	-0.00256344	0.00004952
13500	-13.6204239	0.00238150	0.00003851

From Table 5, we can summarize that for s = 0, the Fourier series diverges. Although, after regularization the value of the Riesz means of order s = 1 of the Fourier series is approximately near to zero but the series diverges. Finally, for s = 2, the series converges.

3.3.2 Regularization of the Two Dimensional Heat Transfer Problem at Critical Index

From Table 5, it is clear that the series converges for s = 2. We know that the series converges when order $s>\frac{N-1}{2}+1$. So, s = 3/2 is the critical point for the regularized series. Here, we choose the critical point s = 1.5. To understand the difference between below the critical point and above the critical point we take two more points 1.5 – ε and 1.5 + ε (ε is very small number). Here, we choose ε = 0.1.Where all the other parameters are kept the same.

Table 6: Solution of the regularized series in equation (40) for different s (s = 1.4, s = 1.5, s = 1.6).

λ	$E_\lambda^{1.4}(1/2, 1/2, 0)$	$E_\lambda^{1.5}(1/2, 1/2, 0)$	$E_\lambda^{1.6}(1/2, 1/2, 0)$
4500	0.00020801	0.00017612	0.00017332
6500	-0.00000105	0.00005941	0.00008501
8500	0.00010912	0.00008002	0.00006922
10500	-0.00004652	0.00000438	0.00002671
10550	0.00002892	0.00003761	0.00004122
10750	-0.00002072	0.00001082	0.00002711
10900	0.00005763	0.00004984	0.00004572
11100	-0.00002301	0.00001051	0.00002701
14500	0.00002222	0.00002484	0.00002722

From Table 6, it is clear that below the critical point, the series diverges and after critical point, the series converges. But, at the critical point, the answer of the regularized series is so near to zero but it diverges.

4. CONCLUSION

In this paper, we studied optimization of the regularization of the Fourier series of the plate problems such as the steady state heat transfer problem and insulated heat transfer problems. Here, we investigated the regularized Fourier series solution at a fixed point on the plate at the initial time and critical index. As it is expected, we achieved the good convergence after the critical point.

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