Nonlinear Systems and Complexity 23
Series Editor: Albert C. J. Luo

Kenan Taş · Dumitru Baleanu
J. A. Tenreiro Machado *Editors*

Mathematical Methods in Engineering

Theoretical Aspects

Springer

Localization of the Spectral Expansions Associated with the Partial Differential Operators

Mathematical Methods in Engineering pp 217-233 | Cite as

- Abdumalik Rakhimov (1) Email author (abdumalik@iium.edu.my)

1. Faculty of Engineering, International Islamic University Malaysia, , Kuala Lumpur, Malaysia

Chapter
First Online: 21 August 2018

Part of the Nonlinear Systems and Complexity book series (NSCH, volume 23)

Abstract

In this paper we discuss precise conditions of the summability and localization of the spectral expansions associated with various partial differential operators. In this we study the problems in the spaces of both smooth functions and singular distributions. We study spectral expansions of the distributions with the compact support and classify the distributions with the Sobolev spaces. All theorems are formulated in terms of the smoothness and degree of the regularizations.

Notes

Acknowledgements

This research work is supported by IIUM FRGS 14 142 0383.

References

1. Alimov, Sh.A.: On spectral decompositions of functions in H_p^α. Mat. Sb. **101**(1), 3–20 (1976)
Google Scholar (https://scholar.google.com/scholar?
q=Alimov%2C%20Sh.A.%3A%20On%20spectral%20decompositions%20of%20functions%20in%20%0A%20%2
0%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%0A%20%20%20%20%20%2
0%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%0A%20%20%20%20%20%20%2
0%20%20%20%20%20%20%20%20%20%20%20%20%0A%20%20%20%20%20%20%20%20%20%20%2
0%20%20%20%20%20%20%20%20%0A%20%20%20%20%20%20%20%20%20%20%20%20%20%20%2
0%20%20%20%20%20%20%0A%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%2
0%20%20%20%20%20%20%0A%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%2
0%20%20%20%20%20%20%20%20%0A%20%20%20%20%20%20%20%20%20%20%20%20%20%20%2
0%20%20%20%20%20%20%20%20%20%20%20%20H%0A%20%20%20%20%20%20%20%20%20%20%20%
20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%0A%20%20%20%20%20%20%20%20%
20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%0A%20%20%20%20%20%20%
20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20p%0A%20%20%20
%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%0A%20
%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20
%0A%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20
%20%20%20%20%CE%B1%0A%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20
%20%20%20%20%20%20%20%20%0A%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20
%20%20%20%20%20%20%20%20%0A%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20
%20%20%20%20%20%20%0A%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20
%20%20%0A%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%20%0A%24

Localization of the spectral expansions associated with the partial differential operators.

Abdumalik Rakhimov

International Islamic University, Kuala Lumpur, Malaysia.
e-mail: abdumalik@iium.edu.my

Abstract. In this paper we prove precise conditions of the summability and equi-summability of the spectral expansions associated with partial differential operators. It is established sharp relations between index of summation and singularity of the distribution.

1. **Introduction**

The solutions of the engineering problems can be obtained with the application of series or transformations depending on the domain, boundary and initial conditions. For example, investigations of the various vibration processes in the real interval (finite, semi-finite or infinite) requires application of the Fourier series and integrals. This type of problems are the main reasons of the significance of the study the questions on convergence of the Fourier series and integrals in different topology.

Multidimensional case, in contrast to the one dimension, is involving various methods of summations of the Fourier series and integrals. Note that some of the summation methods is linked to the spectral theory of the partial differential operators. For example, a spherical partial sums of the multiple Fourier trigonometric series coincides with the spectral expansions associated with the Laplace operator on the torus.

The spectral expansions associated with the elliptic partial differential operators in the spaces of the smooth functions well studied in many papers. But many phenomena in nature require for its description either "bad" functions or even they cannot be described by regular functions. Therefore, one has to deal with the distributions that describe only integral characteristics of phenomena.

Application of the spectral methods in the spaces of distributions, leads to the study of convergence and/or sumability problems of the spectral expansions of the linear continuous functional. We will study convergence and summability problems of the spectral expansions of distributions in the classical means in the domains where they coincide with the regular functions. We prove that the singularities of the distributions will effect negatively to the convergence and/or summability even at regular points. For example, the Fourier trigonometric series of the Dirac function is diverges at a regular point due to the effect of the singularity at zero, although its arithmetic means converge.

2. **Summabilty of the spectral expansions of distributions.**

Let Ω - an arbitrary N - dimensional domain. Consider a polynomial by ξ, $\xi \in R^N$, of order $2m$ with coefficients in $C^\infty(\Omega)$

$$A(x,\xi) = \sum_{|\alpha|\leq 2m} a_\alpha(x)\xi^\alpha, \tag{2.1}$$

where α denotes a multi-index, i.e. N - dimensional vector with non-negative integer components $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N)$, $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_N$ is called the length of the multi-index α, $\xi^\alpha = \xi_1^{\alpha_1}\xi_2^{\alpha}....\xi_N^{\alpha}$, here ξ_j - component of the vector ξ.

Denote

$$D_j = \frac{1}{i}\frac{\partial}{\partial x_i} \quad and \quad D^\alpha = D_1^{\alpha_1}D_2^{\alpha_2}.....D_N^{\alpha_N}.$$

Differential operator

$$A(x,D) = \sum_{|\alpha|\leq 2m} a_\alpha(x)\ D^\alpha, \tag{2.2}$$

is called an elliptic operator of the order $2m$, at the point x, if for any $\xi \in R^N$

$$A_0(x,\xi) = \sum_{|\alpha|=2m} a_\alpha(x)\ \xi^\alpha \geq c(x)|\xi|^{2m},\ c(x) > 0.$$

$A(x,D)$ is called an elliptic operator in the domain Ω if it is elliptic in each point of the domain.

Denote by $C_0^\infty(\Omega)$ the space of all infinite differentiable functions in the domain Ω with the compact support in Ω.

Let A an operator in the Hilbert space $L_2(\Omega)$ with the domain of definition $D(A) = C_0^\infty(\Omega)$, acting as $Au(x) = A(x,D)u(x)$, $u(x) \in C_0^\infty(\Omega)$. Let A is a symmetric operator, i.e. for any u and v from $C_0^\infty(\Omega)$

$$(u,v) = (u,Av).$$

Also suppose that A is semi-bounded which means there is a constant μ such that for any $u \in C_0^\infty(\Omega)$

$$(u,u) \geq \mu(u,u).$$

From the Fredrix theorem [5] it follows that the operator A has at least one self-adjoin extension $\hat{A}$ with the same lower bound μ . By the John von Neumann spectral theorem (see, for instance, in [37]) the operator $\hat{A}$ can be represented as

$$\hat{A} = \int_{\mu}^{\infty} \lambda dE_{\lambda}.$$

where projectors E_λ are integral operators

$$E_\lambda f = \int_\Omega \Theta(x,y,\lambda) f(y) dy, \quad f \in L_2(\Omega). \tag{2.3}$$

The kernel $\Theta(x,y,\lambda)$ is called the spectral function of the operator $\hat{A}$ and the expression (2.3) is called the spectral expansions of f associated with the self-adjoin operator $\hat{A}$.

Define the Riesz means of order $s \geq 0$, of the spectral expansions $E_\lambda f$ as follows

$$E_\lambda^s f = \int_\mu^\lambda \left(1 - \frac{t}{\lambda}\right)^s dE_t.$$

The operators E_λ^s, as well as E_λ, are integral operators with the kernels

$$\Theta^s(x,y,\lambda) = \int_\mu^\lambda \left(1 - \frac{t}{\lambda}\right)^s d\Theta(x,y,t).$$

Note, from the Garding theorem [5] it follows that the function $\Theta(x,y,\lambda)$ belongs $C^\infty(\Omega\times\Omega)$ for each $\lambda > 0$. This allow to define the spectral expansions of the distributions with the compact support.

Denote by $\mathcal{E}'(\Omega)$ - the space of the linear continuous functionals on $C^\infty(\Omega)$. Then for any distribution $f \in \mathcal{E}'(\Omega)$ the Riesz means of its spectral expansions define as follows

$$E_\lambda^s f(x) = < f, \Theta^s(x,y,\lambda) >, \tag{2.4}$$

where the functional f is acting on $\Theta^s(x,y,\lambda)$ with the respect to the second variable. Note that $E_\lambda^s f(x) \in C^\infty(\Omega)$ for any distribution f from $\mathcal{E}'(\Omega)$, $s \geq 0$ and $\lambda > 0$.

The relation (2.4) can be also considered in the classical sense on the domains where f coincides with the locally integrable function.

For any integer ℓ denote the Sobolev spaces $H^\ell(\Omega) = W_2^\ell(\Omega)$ [38].

Theorem 2.1. *Let* $f \in \mathcal{E}'(\Omega) \bigcap H^{-\ell}(\Omega)$, $\ell > 0$. *If* $s \geq (N-1)/2 + \ell$, *then uniformly in any compact* K *from* $\Omega \setminus suppf$

$$\lim_{\lambda \longrightarrow \infty} E_\lambda^s f(x) = 0.$$

From the estimate of the spectral function $\Theta(x, y, \lambda)$ (see [18] theorem 6.1) we get the following lemma.

Lemma 2.2. *Let* Ω_o *be some sub-domain* $\Omega_0 \subset \Omega$ *and let* $f \in \mathcal{E}'(\Omega) \bigcap H^{-\ell}(\Omega)$ *be such that* $suppf \subset \Omega_o$. *Let* K *be a compact set from* $\Omega \setminus \Omega_o$ *and* $s = (N-1)/2 + \ell$. *Then the estimate*

$$|E_\lambda^s f(x)| \leq C \, \|f\|_{-\ell}, \tag{2.5}$$

is valid uniformly with respect to $x \in K$.

The proof of the Theorem 2.1 follows from the Lemma 2.2 and the fact that the space $C_0^\infty(\Omega)$ is dense in $\mathcal{E}'(\Omega) \bigcap H^{-\ell}(\Omega)$.

Theorem 2.3. *Let* $\ell > 0$ *and let* x_0 *a point in the domain* Ω. *If* $s < (N-1)/2 + \ell$, *then there exists a distribution* f *from* $\mathcal{E}'(\Omega) \bigcap H^{-\ell}(\Omega)$, *such that* $x_0 \in \Omega \setminus suppf$ *and*

$$\overline{\lim_{\lambda \longrightarrow \infty}} E_\lambda^s f(x_0) = +\infty.$$

The Theorem 2.3 proves sharpness of the inequality $s \geq (N-1)/2 + \ell$, in the Theorem 2.1. It follows from the estimate of the spectral function from the bottom [18] and the Banach-Steinhaus theorem.

Using the Hermander theorem [19] the Theorem 2.1 can be extended as follows

Theorem 2.4. *Let* $\ell > 0$ *and* Ω_0 *- a sub-domain of* Ω. *If* $s \geq (N-1)/2 + \ell$, *then for any distribution* $f \in (\Omega) \cap \mathcal{E}'(\Omega) \cap H^{-\ell}(\Omega)$, *then uniformly on each compact* K *from* Ω_0

$$\lim_{\lambda \longrightarrow \infty} E_\lambda^s f(x) = f(x).$$

Case $p \neq 2$ is more complicated even for the spectral expansions of the smooth functions. In this case we have the following results [25].

For any real number ℓ by $W_p^{-\ell}(\Omega)$ denote the Sobolev spaces [38].

Theorem 2.5. *Let* $f \in \mathcal{E}'(\Omega) \bigcap W_p^{-\ell}(\Omega)$, , $\ell > 0$, $1 < p \le 2$. *If* $s \ge (N-1)/p + \ell$, *then uniformly in any compact* K *from* $\Omega \setminus suppf$

$$\lim_{\lambda \longrightarrow \infty} E_\lambda^s f(x) = 0.$$

A problem of the sharpness of the condition

$$s \ge (N-1)/p + \ell,$$

in the Theorem 2.5 is complicated. This question is complicated even for the spectral expansions associated with the Laplace operator in the arbitrary domain.

3. **Localization and uniform convergence of the eigenfunction expansions.**

Let Ω be a bounded domain in R^N with the smooth boundary $\partial\Omega$. Let $\hat{A}$ be a self-adjoin extension of a positive formally elliptic differential operator of order $2m$ with the regular boundary conditions ([5]).

Denote by $\{u_n(x)\}$ a complete orthonormal in $L_2(\Omega)$ system of eigenfunctions of the operator $\hat{A}$ corresponding to the sequence of eigenvalues $0 < \lambda_1 < \lambda_2 < .... < \lambda_n \to \infty$. For any function $f \in L_2(\Omega)$ we introduce the Riesz means of order s of the partial sums of the Fourier series as follows

$$E_\lambda^s f(x) = \sum_{\lambda_n < \lambda} \left(1 - \frac{\lambda_n}{\lambda}\right)^s f_n u_n(x). \tag{3.1}$$

Here $\lambda > 0$, $f_n = (f, u_n)$ are the Fourier coefficients of the function f with respect to the system $\{u_n(x)\}$.

Note that if $s = 0$, then (3.1) is just the partial sum of the Fourier series of the function f.

The precise conditions of the uniform convergence of $E_\lambda^s f(x)$ on the compact subsets of the domain Ω are established by V.A. Il'in (see in [15]).

Theorem 3.1. *If*

$$\alpha \ge \frac{N-1}{2}, \quad \alpha p > N, \quad p \ge 1 \tag{3.2}$$

then the Fourier series via the eigenfunctions of the Laplace operator of any function with compact support belonging to the Sobolev space $W_p^\alpha(\Omega)$ *converges uniformly on any compact subset of the domain* Ω.

Convergence of the Riesz means (3.1) of the smooth functions on the compact subsets of the domain Ω requires a modification of the condition (3.2) in the Theorem 1.1 as follows

$$\alpha + s \geq \frac{N-1}{2}, \quad \alpha p > N, \quad s \geq 0, \quad p \geq 1. \tag{3.3}$$

The sharpness of the first inequality in (3.3) for the eigenfunction expansions associated with the Laplace operator is proved by V.A. Il'in (see in [15]). The preciseness of the second inequality in (1.3) follows from the fact that the condition $\alpha p \leq N$ implies the existence of an unbounded function with the compact support belonging to the appropriate Sobolev space for which its Fourier series cannot converge uniformly.

Moreover, the conditions (3.3) are sufficient for the functions in the Nikol'skii spaces $H_p^\alpha(\Omega)$. The last statement is proved in the case of expansions associated with the eigenfunctions of the Laplace operator by V.A. Il'in and Sh.A. Alimov [14], in the case of the expansions associated with the elliptic operators of the second order with the variable coefficients by V.A. Il'in and E.I. Moiseev [13]. Finally, for the general elliptic differential operators of order $2m$ Sh.A. Alimov is proved in [2] the following statement

Theorem 3.2. *If f belongs to the space $\overset{\circ}{H}{}_p^\alpha(\Omega)$ and has the compact support in Ω, then under the conditions (3.3) the Riesz means $E_\lambda^s f$ converge as $\lambda \to +\infty$ to f uniformly on any compact $K \subset \Omega$.*

Here $\overset{\circ}{H}{}_p^\alpha(\Omega)$, $(\overset{\circ}{W}{}_p^\alpha(\Omega))$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm of the Nikol'skii (Sobolev) space $H_p^\alpha(\Omega)$ $(W_p^\alpha(\Omega))$.

In the case in which the second condition in (3.3) is replaced by $\alpha p = N$, it is necessary to assume that the function f is continuous (see [1]).

Theorem 3.3. *Let Ω_0 be an arbitrary open subset of Ω and let*

$$\alpha + s > \frac{N-1}{2}, \quad \alpha p = N, \quad s \geq 0, \quad p \geq 1. \tag{3.4}$$

Then for any function $f \in \overset{\circ}{W}{}_p^\alpha(\Omega)$ continuous on Ω_0

$$\lim_{\lambda \to \infty} E_\lambda^s f(x) = f(x). \tag{3.5}$$

uniformly on any compact set $K \subset \Omega_0$.

The first condition $\alpha + s > \frac{N-1}{2}$ in (3.4) is also precise [1].

Theorem 3.4. *Let* x_0 *be an arbitrary point of the domain* Ω *and let*

$$\alpha + s = \frac{N-1}{2}, \quad \alpha p = N, \quad s \geq 0, \quad p \geq 1. \tag{3.6}$$

Then there exists a function $f \in \mathring{W}_p^\alpha(\Omega)$, *which is continuous in* Ω, *and such that*

$$\overline{\lim_{\lambda \to \infty}} E_\lambda^s f(x_0) = +\infty. \tag{3.7}$$

These results are extended to the Nikol'skii spaces in [21].

4. Summability of the multiple Fourier series of the periodic distributions.

We denote by $C^\infty(T^N)$ the space of 2π periodic in each variable, infinitely differentiable on R^N functions, where $T^N = \{x \in R^N : -\pi < x_j \leq \pi\}$.

Let $\gamma = (\gamma_1, \gamma_2, ...\gamma_N)$ denotes a multi-index. The system of the semi-norms $Sup_{x \in T^N} |D^\gamma f(x)|$ produces a locally convex topology in $C^\infty(T^N)$, where γ runs over the set of all multi-indexes. We denote by $\mathcal{E}(T^N)$ corresponding locally convex topological space. Let $\mathcal{E}'(T^N)$ is the space of the periodic distributions, i.e. the space of the continuous linear functionals on $\mathcal{E}(T^N)$.

For any distribution f from $\mathcal{E}'(T^N)$ we define its Fourier coefficients f_n as the action of the distribution f on the test function $(2\pi)^{-\frac{N}{2}} \exp(-inx)$, where $x \in T^N$ and $n \in Z^N$ is N dimensional vector with integer coordinates. Then f can be represented by the Fourier series

$$f \ = \ (2\pi)^{-\frac{N}{2}} \sum_{n \in Z^N} f_n \ \exp(inx), \tag{4.1}$$

which always converges in the weak topology (see, for example, in [23]).

Consider the following polynomial :

$$P_m(n) = \left(\sum\nolimits_{j=1}^{r+1} n_j^2\right)^{m+1} + \left(\sum\nolimits_{j=r+2}^{N} n_j^2\right)^m \left(\sum\nolimits_{j=1}^{N} n_j^2\right), \tag{4.2}$$

where $n = (n_1, n_2, ..., n_N) \in Z^N$, m is a positive integer number and $r = 0, 1, ..., N-1$.

The polynomial $P_m(n)$ is a homogeneous of degree $2(m+1)$, i.e.

$$P_m(\lambda \cdot n) = \lambda^{2(m+1)} \cdot P_m(n)$$

and an elliptic, i.e.

$$P_m(n) > 0, \ n \neq 0.$$

Thus a family of bounded sets

$$\Lambda(\lambda) \ = \ \{n \in Z^N : P_m(n) < \lambda\}, \ \lambda \in R^+\}$$

enjoying the following properties:

a) for any pairs $(\lambda_1, \lambda_2) \in R^+ \times R^+$ there is $\lambda \in R^+$, such that $\Lambda(\lambda_1) \cup \Lambda(\lambda_2) \subset \Lambda(\lambda)$.

b) $\bigcup_{\lambda \in R^+} \Lambda(\lambda) = Z^N$.

Let $f \in \mathcal{E}'(T^N)$. Then Λ - partial sums of series (4.1) define by equality

$$E_\lambda f(x) \ = \ (2\pi)^{-\frac{N}{2}} \sum_{\Lambda(\lambda)} f_n \ \exp(in \cdot x). \tag{4.3}$$

For any real s , $s \geq 0$, we define the Riesz means of (4.3) by

$$E_\lambda^s f(x) \ = \ (2\pi)^{\frac{-N}{2}} \sum_{\Lambda(\lambda)} \left(1 - \frac{P_m(n)}{\lambda}\right)^s f_n \ \exp(inx). \tag{4.4}$$

At $s = 0$ we obtain the partial sums (4.3).

Summability of the series (4.1), as well as its regularization (4.4), depends on the power of singularity of f. In order to classify singularities of distributions, we apply the periodic Liouville spaces $L_p^\alpha(T^N)$, $1 < p \leq \infty$, $\alpha \in R$ [38].

Theorem 4.1. Let $f \in L_p^{-\alpha}(T^N) \bigcap \mathcal{E}'(T^N)$, $1 < p \leq 2$, $\alpha > 0$, and coincides with zero in $\Omega \subset T^N$. If

$$s > max\{\frac{(N-r-1)(1-\frac{1}{2m})}{p} \ + \ \frac{r}{2}, \ \frac{N-1}{2}\} \ + \ \alpha \tag{4.5}$$

then uniformly on any compact set $K \subset \Omega$

$$\lim_{\lambda \longrightarrow \infty} E_\lambda^s f(x) = 0.$$

The Riesz means (4.4) can be written as

$$E_\lambda^s f(x) \ =< f, \ D_\lambda^s(x-y) >, \tag{4.6}$$

where f acts on $D^s_\lambda(x-y)$ by y and $D^s_\lambda(x)$ is the Riesz means of $\Lambda-$ partial sums of the multiple Fourier series of the Delta function:

$$(4.7)\qquad D^s_\lambda(x) \ = \ (2\pi)^{\frac{-N}{2}} \sum_{\Lambda(\lambda)} \left(1-\frac{P_m(n)}{\lambda}\right)^s \ \exp(inx).$$

If $r=N-1$, then $D^s_\lambda(x)$ is exactly the Riesz means of the Dirichlet kernel [35].

First, we estimate (4.7) in the norm of the positive Liouville spaces. In this, we use the relation between the kernel (4.7) and the relevant kernel of Fourier integrals. Such a relation known as the Poisson summation formula. The kernel for the corresponding Fourier integrals can be described by the same polynomial P_m replacing its argument range from $n\in Z^N$ to $\xi \in R^N$:

$$(4.8)\qquad \Theta^s_\lambda(x) \ = \ (2\pi)^{\frac{-N}{2}} \int_{\Lambda(\lambda)} \left(1-\frac{P_m(\xi)}{\lambda}\right)^s \ \exp(i\xi\cdot x)d\xi,$$

where in the definition of the domain $\Lambda(\lambda)$ its range must be changed accordantly.

The following asymptotic formula is valid for the kernel (4.8) [9]:

Lemma 4.2. Let $x\in R^N$, $x=(x',x'')$, $x'\in R^{r+1}$, $x''\in R^{N-r-1}$, $0<\delta_0<|x'|$, $\mu=\lambda^{\frac{1}{2(m+1)}}$. Then for $|x''|<\varepsilon\mu^{-(1-\frac{1}{2m})}$, $0<\varepsilon<\frac{1}{2}$, and $\mu\longrightarrow\infty$
(4.9)

$$\Theta^s_\lambda(x) \ = \ \frac{c\mu^N \cos(\mu|x'|+(\frac{r}{2}-s)\frac{\pi}{2})}{(\mu|x'|)^{\frac{r+2}{2}+s+\frac{N-1-r}{2m}}} \times (\ 1+ \ O(\frac{1}{\mu|x|}) \ + \ O(|x''|\mu^{1-\frac{1}{2m}}))$$

Using the Poisson summation formula the following relation between two kernel can be established

$$(4.10)\qquad D^s_\lambda(x) \ = \ \Theta^s_\lambda(x) \ +\Theta^s_{*,\lambda}(x),$$

where $\Theta^s_{*,\lambda}(x)$ defined as

$$(4.11)\qquad \Theta^s_{*,\lambda}(x) \ = \ \sum_{n\in Z^N,\ n\neq 0} \Theta^s_\lambda(x+2\pi n).$$

Then from Lemma 4.2 immediately obtain the following

Lemma 4.3. Let $\varepsilon > 0$ an arbitrary small number and $|x_i| \leq 2\pi - \varepsilon$, for any $i = 1, 2, 3, ..., N$. If s satisfies *(4.5),* then

$$\Theta^s_{*,\lambda}(x) \;=\; O(\lambda^{\frac{1}{2(m+1)}})^{N-s-1-\frac{r}{2}-\frac{N-1-r}{2m}} \tag{4.12}$$

Lemma 4.3 provides an estimate of the second term in the right-hand side of (4.11). Moreover, if $0 < \delta_0 < |x'|$, then from (4.9) we obtain an estimation for the first term in (4.11). Thus, we proved the following

Lemma 4.4. Let $\varepsilon > 0$ an arbitrary small number and $|x'| > \varepsilon$. If s satisfies *(4.5)* , then

$$D^s_\lambda(x) \;=\; O(\lambda^{\frac{1}{2(m+1)}})^{N-s-1-\frac{r}{2}-\frac{N-1-r}{2m}} \tag{4.13}$$

We have the following estimation (see [18], [19]):
let $K \subset\subset T^N$ *a compact set, then uniformly by* $x \in K$

$$\|D^s_\lambda(x-y)\|_{L_2(F)} \;=\; O(\lambda^{\frac{N-1-2s}{4(m+1)}}) \;, \tag{4.14}$$

where F *an arbitrary domain in* T^N *such that* $\overline{F} \bigcap K = \emptyset$.

Then using the Stein interpolation theorem for the analytical family of the linear operators ([33], [34]) with $q = \infty$ (Lemma 4.4) and $q = 2$ (estimation (4.14)), obtain the following estimate the kernel $D^s_\lambda(x)$ in the norm of $L_q(T^N)$

Lemma 4.5. Let s satisfies *(4.5)* and $K \subset\subset T^N$ an arbitrary compact set. Then uniformly by $x \in K$

$$\|D^s_\lambda(x-y)\|_{L_q(F)} \;=\; O(\lambda^{\frac{1}{2(m+1)}})^{N-s-1-\frac{r}{2}-\frac{N-1-r}{2m}}, \tag{4.15}$$

where F an arbitrary domain in T^N such that $\overline{F} \bigcap K = \emptyset$, $2 \leq q \leq \infty$.

Let a distribution f has a compact support and belongs to the space $L^{-\alpha}_p(T^N)$, where $1 < p \leq 2$, $\alpha > 0$. Let K an arbitrary compact set from $T^N \backslash suppf$ and s satisfies (4.5) . Then from (4.6) get the following

$$|E^s_\lambda f(x)| \;\leq\; \|f\|_{-\alpha,p} \, \|D^s_\lambda(x-y)\|_{\alpha,q,F}, \tag{4.16}$$

where $\| \, \|_{-\alpha,p}$ means a norm in the space $L^{-\alpha}_p(T^N)$ and $\| \, \|_{\alpha,q,F}$ means a norm in the space $L^\alpha_q(F)$, $\frac{1}{q} = 1 - \frac{1}{p}$ and $suppf \subset F$ such that $\overline{F} \bigcap K = \emptyset$.

Then from (4.16) and the Lemma 4.5 it follows

$$E_\lambda^s f(x) = O(1)\ \|f\|_{-\alpha,p} \tag{4.17}$$

uniformly by x from K . Then from (4.17) we get the statement of the Theorem 4.1.

5. Spherical partial sum of the multiple Fourier series and equiconvergene with the Fourier integral

For any distribution $f \in \mathcal{E}'(T^N)$ and any real number $s \geq 0$ the Riesz means of order s of the spherical partial sums of the series (4.1) is defined by

$$\sigma_\lambda^s f(x) \ = \ (2\pi)^{\frac{-N}{2}} \sum_{|n|<\lambda} \left(1 - \frac{|n|^2}{\lambda^2}\right)^s f_n \ \exp(inx). \tag{5.1}$$

where f_n is the value of the functional f on a "test" function $(2\pi)^{-\frac{N}{2}} \exp(-iy\xi)$. If $s = 0$ and $f = \delta$, then from (5.2) we obtain the Direchlet kernel [4].

Let's extend the distribution f to $\mathcal{R}^N$ as

$$g \ = \left\{ \begin{array}{c} f \quad in\ T^N\ , \\ 0\ ,\ in\ \mathcal{R}^N \setminus T^N. \end{array} \right. \tag{5.2}$$

Note that the distribution g belongs to the space $\mathcal{E}'(\mathcal{R}^N)$. Denote by $\hat{g}$ its Fourier transformation. For example, for the Delta function we have $\hat{\delta}(x) = 1$. Then the Bochner-Riesz means of order s of the Fourier integral of the Delta function is

$$\Theta_\lambda^s(x) \ = \ (2\pi)^{\frac{-N}{2}} \int_{|y|<\lambda} \left(1 - \frac{|y|^2}{\lambda^2}\right)^s \ \exp(iy \cdot x)dy. \tag{5.3}$$

Then we can define the Riesz means of the spherical partial sums of the Fourier integral for any distribution $g \in \mathcal{E}'(\mathcal{R}^N)$ as follows

$$R_\lambda^s g(x) \ = < g,\ \Theta_\lambda^s(x-y) > . \tag{5.4}$$

where g is acting on $\Theta_\lambda^s(x-y)$ with the respect to the variable y.

In the critical index $s = \frac{N-1}{2}$, Bochner [10], proved that the localization for (5.4) holds and at the same time it fails for the partial sum (5.1) in the class L_1. He also proved that the localization in the critical index is valid for both expansions in L_2. This result for the expansions in eigenfunctions of the Laplace operator proved by Levitan [17]. Below critical index $\frac{N-1}{2}$ the problem studied by Il'in [12].

Summability of the spectral expansions of distributions studied in [3]. In [3] Sh.A. Alimov obtained precise conditions of the localization of the spectral expansions associated with the Laplace operator. These questions for the Fourier series studied in [23] and for the Forier integral studied in [24].

Theorem 5.1. *Let* $\ell > 0$ *and* $s = \frac{N-1}{2} + \ell$. *Then for any* $f \in L_p^{-\ell}(T^N)$ *with* $1 < p \leq 2$ *and* $suppf \subset \Omega \subset\subset T^N$

$$\sigma_\lambda^s f(x) \;=\; R_\lambda^s F(x) \;+\; O(1)\; \|f\|_{-\ell,p},$$

where $x \in T^N \setminus \overline{\Omega}$ *and* $\|\cdot\|_{-\ell,p}$ *a norm in* $L_p^{-\ell}(T^N)$:

$$\|f\|_{-\ell,p} \;=\; (2\pi)^{-\frac{N}{2}}\; \|(1+|n|^2)^{\ell/2}\; f_n\; \exp(inx)\|_p.$$

In case $s < \frac{N-1}{2} + \ell$ the statement of the Theorem 5.1 is not valid for any distribution [23]. In case $p = 2$ the Theorem 5.1 is proved in [24].

Let $\Theta_{*,\lambda}^s(x)$ denotes the following

$$\Theta_{*,\lambda}^s(x) \;=\; \sum_{n \in Z^N, n \neq 0} \Theta_\lambda^s(x + 2\pi n). \tag{5.5}$$

Then we get

Lemma 5.2. *Let* $\ell > 0$, $s = \frac{N-1}{2} + \ell$. *Then uniformly on any compact set* $K \subset T^N$

$$|\Theta_{*,\lambda}^s(x)| \;=\; O(\lambda^{-\frac{\ell}{4}}). \tag{5.6}$$

As in previous section using the Poisson formula of summation we get the following relation between expansions (5.1) and (5.4)

$$\sigma_\lambda^s f(x) \;-\; R_\lambda^s F(x) \;= < f, \Theta_{*,\lambda}^s(x-y) > . \tag{5.7}$$

Let $\Omega_0 \subset\subset \Omega$ and *suppf* $\subset \Omega_0$. Then from the Cauchy-Schwartz inequality, taking into account that $f \in L_p^{-\ell}(T^N)$, obtain the following

$$| < f, \Theta_{*,\lambda}^s(x-y)| \;\leq\; \|f\|_{-\ell,p} \|\Theta_{*,\lambda}^s(x-y)\|_{\ell,p,0} \tag{5.8}$$

where $\|\Theta_{*,\lambda}^s(x-y) > \|_{\ell,p,0}$ is a norm of $\Theta_{*,\lambda}^s(x-y)$ in $L_p^\ell(\Omega_0)$ via $y \in \Omega_0$,

From (5.8) and the Lemma 5.2 we get the following

Lemma 5.3. *Let* $s = \frac{N-1}{2} + \ell,\ \ell > 0, \quad f \in L_p^{-\ell}(T^N) \cap \mathcal{E}'(T^N),\ 1 < p \le 2$ *and let* $suppf \subset \Omega \subset\subset T^N$.

Then uniformly on any compact set $K \subset T^N \setminus \overline{\Omega}$

$$< f, \Theta^s_{*,\lambda}(x-y) >= \mathcal{O}(1)\|f\|_{-\ell,p}$$

Then the statement of the Theorem 5.1 follows, in the standard way, from the Lemma 5.3. This statement is proved in [24] in case $p = 2$ and in [32] for any p.

6. **Uniform convergence on closed domains.**

The uniform convergence of the Fourier series on the closed domains $\overline{\Omega}$ was studied by V.A. Il'in (see [16]). In [16] for the eigenfunction expansions associated with the first, second and third boundary conditions for the Laplace operator it was proved that if $f \in W_p^{\frac{N+2}{2}}, p > \frac{2N}{N-1}$ and the functions $f, \Delta f, ..., \Delta^\beta f$, up to a certain order β, satisfy the appropriate boundary conditions, then the Fourier series of f converges uniformly on the closed domain $\overline{\Omega}$.

For the elliptic differential operator of order $2m$ with the regular boundary conditions G.I. Eskin (see [11]) proved that the eigenfunction expansion of a function in $\overset{\circ}{W}{}_p^{\frac{2N-1}{4}+\varepsilon}$ with any $\varepsilon > 0$ converges uniformly on the closed domains.

E.I. Moiseev studied the problem for the elliptic operators of second order for the first boundary value problem. In [22] it is proved that if f is a function with compact support in the space $W_p^{\frac{N-1}{2}}, p > \frac{2N}{N-1}$, such that the series

$$\sum_{n=1}^{\infty} \lambda_n^{\frac{N-1}{2}} (\ln \lambda_n)^{2+\varepsilon} f_n^2$$

converges, then its expansion via eigenfunctions converges uniformly on the closed domain $\overline{\Omega}$.

Moreover, it was proved in [22] that the following estimate

$$\sum_{|\sqrt{\lambda_n}-\mu|\le 1} u_n^2(x) = O(\mu^{N-1}\ln^2\mu) \tag{6.1}$$

is valid uniformly on the closed domain $\overline{\Omega}$.

In [6] uniform convergence of expansions via eigenfunctions of the elliptic differential operator of order $2m$ with the Lopatinsky boundary condition was studied and the following result is proved.

Theorem 6.1. *Let f be an arbitrary continuous function with compact support in Ω. Then the Riesz means $E_\lambda^s f(x)$ of order $s > \frac{N}{2}$ converge to f uniformly on the closed domain $\overline{\Omega}$.*

In [26] by using estimate (2.1) the condition $s > \frac{N}{2}$ in the Theorem 6.1 was replaced by $s > \frac{N-1}{2}$.

We mention also the following result (see [27]).

Theorem 6.2. *Let*

$$\alpha + s > \frac{N-1}{2}, \quad \alpha p \geq N, \quad s \geq 0, \quad p \geq 1. \tag{6.2}$$

Then for any continuous function $f \in \overset{\circ}{H}{}_p^\alpha(\Omega)$ with the compact support in the domain Ω uniformly on $\overline{\Omega}$

$$\lim_{\lambda\to\infty} E_\lambda^s f(x) = f(x). \tag{6.3}$$

Note, from the Theorem 3.4 it follows that in the case $\alpha p = N$ the condition $\alpha + s > (N-1)/2$ is precise. In the case $\alpha p > N$ this problem is open. The question of the summability on the closed domain in the spaces of the distributions studied in [7].

Now we consider the problem of convergence of expansions via eigenfunctions in the spaces with mixed norm.

The space of all measurable functions with finite norm

$$\|f\|_{L_{pq}(\mathcal{R}^N)} = \| \|f\|_{L_p(\mathcal{R}^k)} \|_{L_q(\mathcal{R}^{N-k})}$$

is called the space with mixed norm $L_{pq}(\mathcal{R}^N)$. If a function is defined in the domain Ω then the corresponding space can be defined by extending a function by zero outside of the domain Ω.

By H_{pq}^α we denote the Banach space of all measurable functions with respect to the norm

$$\|f\|_{H_{pq}^\alpha(\Omega)} = \|f\|_{L_{pq}(\Omega)} + \sum_{|k|=\ell} \sup_z |z|^{-\kappa} \|\Delta_z^2 \partial^k f(y)\|_{L_{pq}(\Omega_{|z|})}.$$

Here $\alpha = \ell + \kappa$, ℓ is a non negative integer, $0 < \kappa \leq 1$, $p, q \geq 1$, $k = (k_1, k_2, ..., k_n)$ multiindex. $|k| = k_1 + k_2 + ... + k_n$ and $\partial^k f$ denotes the weak derivative

$$\partial^k f(y) = \frac{\partial^{|k|} f(y)}{\partial y_1^{k_1}, \partial y_2^{k_2}, ..., \partial y_n^{k_n}}.$$

The symbol $\Delta_z^2 \partial^k f(y)$ denotes the second difference of the function $\partial^k f(y)$:

$$\Delta_z^2 \partial^k f(y) = \partial^k f(y+z) - 2\partial^k f(y) + \partial^k f(y).$$

$\|f\|_{L_{pq}(\Omega)}$ denotes the norm in the space L_{pq} and, for $h > 0$, $\Omega_h = \{x \in \Omega : dist(x, \partial\Omega) > h\}$.

By $\overset{\circ}{H}{}^{\alpha}_{pq}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ with respect to the norm of the space $H^\alpha_{pq}(\Omega)$.

Using the methods of [1]-[2] for functions in the spaces with the mixed norm appropriate theorems on convergence of the spectral expansions associated with the Laplace operator on compact subsets of the domain were obtained in [36].

Theorem 6.3. *Let $f(x)$ be a continuous function with compact support in the domain Ω belonging to the space $\overset{\circ}{H}{}^{\alpha}_{pq}(\Omega)$ and*

$$\text{(6.4)} \qquad \alpha > \frac{N-1}{2} - s, \quad \alpha = \frac{N-k}{q} + \frac{k}{p}, \quad 2 \le p < q, \quad 0 < k < N.$$

Then uniformly on $\overline{\Omega}$

$$\lim_{\lambda \longrightarrow \infty} E^s_\lambda f(x) = f(x).$$

This theorem in the spaces of distributions from the Sobolev spaces with the mixed norm is proved in [29].

Note that Theorems 6.1 and 6.2 are obtaned only for the eigenfunction expansions associated withe first boundary problem for the Laplace operator. Analogue of the theorem 6.2 in the generalized Sobolev spaces of distributions is proved in [7]. Recently in [30] analogue of Theorem 6.2 in the generalized Sobolev spaces of distributions is proved for the eigenfunction expansions associated with the Navie boundary conditions for the bi-harmonic operator.

The problems of the convergence/summability of the Fourier series (spectral expansions) on the closed domains remain open for any other boundary conditions than first boundary condition (including second and third type boundary conditions) even for the Laplace operator and other operators then the Laplace operator with any boundary conditions.

7. Spectral expansions associated with the operators with singular coefficients.

In this section we consider the Schrodinger operator $L = -\Delta + q(x)$ with the domain $C_0^\infty(R^N)$, where $q(x)$ is potential with singularity at 0 satisfies following conditions

$$|\frac{\partial^{|\alpha|}q(x)}{\partial^{\alpha_1}x_1\partial^{\alpha_2}x_2.....\partial^{\alpha_N}x_N}| \le \; const\; |x|^{-1-\alpha},$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N)$ is a multi-index.

From the Kato-Rellich theorem (see in [37], page 185) it follows that operator L is essentially self-adjoin and bounded from the bottom with some constant μ.

Denote by $\hat{L}$ its only self-adjoin extension (closure) in $L_2(R^N)$.

Let $\{E_\lambda\}$ be the corresponding spectral decomposition of unity. It is well known that the operators E_λ are integral operators whose kernels $\Theta(x, y, \lambda)$ belong to the $C^\infty(R^N)$ with respect to the both variables x and y for any λ. The spectral decomposition of an arbitrary function $g \in L_2(R^N)$ is defined by the formula

$$E_\lambda g(x) = \int\limits_{R^N} g(y)\Theta(x, y, \lambda)dy.$$

Let $f \in \mathcal{E}'(R^N)$. Since $\Theta(x, y, \lambda) \in C^\infty(R^N \times R^N)$, it follows that one can define the spectral decomposition of f by the formula

$$E_\lambda f(x) = \langle f, \Theta(x, y, \lambda)\rangle,$$

where the functional f acts on $\Theta(x, y, \lambda)$ with respect to the second argument.

For any $s \ge 0$, we introduce the Riesz means of the spectral decomposition of f by the formula

$$E_\lambda^s f(x) = \langle f, \Theta^s(x, y, \lambda)\rangle,$$

where $\Theta^s(x, y, \lambda)$ is the Riesz mean of order s of the spectral function,

$$\Theta^s(x, y, \lambda) = \int\limits_\mu^\lambda \left(\frac{\lambda - t}{\lambda - \mu}\right)^s d_t\Theta(x, y, t).$$

Theorem 7.1. *Let $\ell > 0, s \ge 0$, and $f \in \mathcal{E}'(R^N) \cap W_2^{-l}(R^N)$. If $s \ge (N - l)/2 + \ell$, then*

$$\lim_{\lambda\to\infty} E_\lambda^s f(x) = 0,$$

uniformly with the respect to $x \in K$ for any compact subset $K \subset R^N \setminus supp(f)$. .

Corollary 7.2. *Let* $f \in \mathcal{E}'(R^N) \cap W_2^{-\ell}(R^N), \ell > 0$, *and let the distribution* f *coincide with a continuous function* $g(x)$ *in a domain* D. *If* $s \geq (N-l)/2 + \ell$, *then*

$$\lim_{\lambda \to \infty} E_\lambda^s f(x) \; = \; g(x)$$

uniformly on each compact set $K \subset D$.

Spectral expansions, associated with the Schrodinger operator is studied by A.R. Khalmukhamedov in various functional spaces (see in [20]). In [28] localization problem of expansions via eigenfunctions of the Schrodinger operator in the bounded domain in the spaces of distributions is studied and the sharp conditions are established. The summability problems of the eigenfunction expansions connected with one Schrodinger operator on the closed domain with the smooth boundary studied in [31].

8. ACKNOWLEDGEMENTS

This research work is supported by IIUM FRGS 14 142 0383.

References

[1] Sh.A. Alimov, *On expanding continuous functions from Sobolev classes in eigenfunctions of Laplace operator,* Sib. Mat. Zh. 19 (1978), no.1, 721-734.

[2] Sh.A. Alimov, *On spectral decompositions of functions in* H_p^α, Mat. Sbornik, USSR, 101 (1976), no. 1, 3-20.

[3] Sh.A. Alimov, On the spectral decompositions of distributions, Doclady Math, 331(6), (1993), 661-662.

[4] Sh.A. Alimov, R.R. Ashurov, A.K. Pulatov, Multiple Fourier series and Fourier integrals, in: Commutative Harmonic Analysis IV, Springer-Verlag, (1992), pp. 1-97.

[5] Sh. A. Alimov, V. A. Il'in, E. M. Nikishin, *Convergence problems of multiple trigonometric series and spectral decompositions. I*, Uspekhi Mat. Nauk 31:6 (192) (1976), 28-83.

[6] Sh.A. Alimov , A.A. Rakhimov, *On the uniformly convergence of spectral expansions in a closed domain.* Dokl. Acad Nauk UzSSR, 10 (1986), 5-7.

[7] Sh.A Alimov, A.A. Rakhimov. On the localization spectral expansions in a closed domain. J. Differential equations, 33(1), (1997), 80-82.

[8] Sh.A Alimov, A.A. Rakhimov. On the localization spectral expansions of distributions. J. Differential equations, 32(6), (1996), 798-802.

[9] R.R. Ahsurov, Localization conditions for spectral decomposition related to eliptic operators with constant coefficients, Math Notes, 33(6), (1983), 847-857.

[10] Bochner S 1936 Trans. Amer. Math. Soc. 40 175-270.

[11] G.I. Eskin, *Asymptotic near the boundary of spectral functions of elliptic self-adjoin boundary problems* Izrael J. Math., 22 (1975), no. 3-4, 214-246.

[12] Il'in V A 1995 Spectral Theory of Differential Operators: self-adjoint differential operators (New York: Consultants Bureau)

[13] V.A. Il'in, E.I. Moiseev, *On spectral expansions connected with with non-negative self-adjoin extension of the second order elliptic operator.* Dokl. Akad. Nauk SSSR. 191 (1971), no. 4,770-772.

[14] V.A. Il'in, Sh.A. Alimov, *The conditions for convergence of the spectral expansions, related to self adjoint extensions of the elliptic operators*, Differential Equations. 7 (1971), no. 4, 670-710.
[15] V.A. Il'in, *Spectral theory of the differential operators,* Nauka, Moscow, 1991. 369 pp. (in Russian).
[16] V.A. Il'in, *On the uniformly convergence eigenfunction expansions associated with Laplace operator in a closed domain.* Mat. Sbornik, 45 (1958), no. 2, 195-232.
[17] Levitan B M 1954 Mat. Sb. 35(77:2) 267-316.
[18] L. Hormander, The spectral function of an elliptic operator. Acta Math. 121, No. 3-4, (1968), 193-218, Zbl. 164,132,
[19] L. Hormander, On the Riesz means of spectral functions and eigenfunction expansions for elliptic differential operators, in: Recent Advances in the Basic Sciences, Proc. Annual Sci. Conf. Belfer Grad. School Sci. 2, Yeshiva Univ., New York, 1965-1966, pp. 155-202.
[20] A.R. Khalmukhamedov., On the negative powers of singular Schrodinger operator and convergence of the spectral expansions. J. Math. Notes 1996. v.56. N.3. 428-436.
[21] N.N. Kozlova, *On the Riesz summability of continuous functions from the Nikol'skii spaces,* Differential equation, 20 (1984), no. 1 46-56.
[22] E.I. Moiseev, *The uniform convergence of certain expansions in a closed domain*, Dokl. Akad. Nauk SSSR., 233(1977), no. 6, 1042-1045.
[23] A.A. Rakhimov, On the localization of multiple trigonometric series of distributions, (2000) Doklady Math. 62(2) 163-165
[24] A.A. Rakhimov, On the equiconvergence of the Fourier series and the Fourier integral of distributions, 2016 AIP Journal Conf. Proc. 1739 020060 http://dx.doi.org/10.1063/1.4952540 1-6
[25] A.A. Rakhimov. Spectral expansions of distributions from negative Sobolev spaces. J. Differential equations, 32(7), (1996), 1011-1013.
[26] A.A. Rakhimov, *On uniform convergence of spectral resolutions of a continuous function in a closed domain.* Izv. Akad. Nauk UzSSR, Ser. Fiz.-Mat. Nauk, 6 (1987), 17-22.
[27] A.A. Rakhimov, *On the uniform convergence of spectral expansions of continuous functions from the Nikol'skii space in the closed domain,* Dep. VINITI. 1564-B87 (1987), 18 pp.
[28] A.A. Rakhimov Localization of the spectral decompositions of distributions, connected with the Schrodinger operator. "Modern problems of math. Physics and information technologies", Tashkent, 2003, V. 1, pp. 167-172.
[29] A.A. Rakhimov On the Fourier series of continuous functions, Proceedings of USA-Uzb Conference on Mathematical Physics, Californian State University, Fullerton, May, 2014, p.1-3.
[30] A.A. Rakhimov, Siti Nor Aini Mohd Aslam Uniform Convergence of the Eigenfunction Expansions Associated with the Polyharmonic Operator on Closed Domain, Proceedings of ICMAE2017, p. 72-76.
[31] A.A.Rakhimov , Kamran Zakariya and Nazir Alikhan, Uniform convergence spectral expansions connected with Schrodinger's operator of continuous function in a closed domain. J. The Nucleus, 2010, V.47, No 4, p.261-265.
[32] A.A.Rakhimov, Torla Bin Hj Hassan1, Ahmad Fadly Nurullah bin Rasedee, On Equiconvergence of Fourier Series and Fourier Integral, IOP Conf. Series: Journal of Physics: Conf. Series 819 (2017) 012025 doi:10.1088/1742-6596/819/1/012025
[33] E. Stein, G. Weiss, Introduction to Fourier Analysis on Euclidean Spaces, Princeton (1971).
[34] E.M. Stein, Interpolation of linear operators, Trans. Amer. Math. Soc. 83 (1956) 482-492.
[35] E.M. Stein, Localization and summability of multiple Fourier series, Acta Math. 100 (1958) 93-147.
[36] V.G. Sozanov, *Uniform convergence and Riesz summability of spectral resolutions*, Mat Zametki, 29 (1981), no.6 887-894.
[37] Reed, M., and Simon B., Modern Methods Matematical Physics, 1980. Volume 2, Acedemics Press.
[38] H. Triebel, Interpolation Theory, Function Spaces, Differential Operators, Berlin, (1978).