Mixed Convection Casson Fluid Flow over an Exponentially Stretching Sheet with Newtonian Heating Effect

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Abstract

This paper deals with mixed convection of Casson fluid which flows over a heated surface that has been stretched exponentially. The governing equations that govern the fluid flow are reduced to ordinary differential equations by imposing suitable similarity variables. Numerical computational was carried out to solve for the f "(0) and θ (0) for some arbitrary values of the mixed convection parameter λ, Biot number Bi and Newtonian fluid parameter β when Pr =7.

Keywords : Mixed Convection, Newtonian Heating, Non-Newtonian Fluid

I. Introduction

The science of non-Newtonian fluid flow over stretching sheet has gained enormous highlight due to its occurrence in many industry and manufacturing processes. The nature of the stretching depends on the desired final output. Sometimes, the sheet either be stretched or shrunk. Many initiatives have been taken into study; i.e sheet being stretched linearly, exponentially, quadratically and etc. Nadeem et al. (2012) considered magnetohydrodynamic (MHD) boundary layer flow of a Casson fluid over an exponentially permeable shrinking sheet which is later extended by Reddy (2016) who investigated convective boundary layer flow of Casson fluid flow past an exponentially inclined permeable stretching surface. Nonlinear stretching sheet of thin flim flow of Casson fluid was considered by Singh and Dandapat (2015) with uniform magnetic field effect.

The nature of mixed convection which encountered in industrial applications has catches attentions among researchers. Mixed convection stagnation-point flow of an incompressible Casson fluid over a stretching sheet under convective boundary conditions was considered by Hayat et al. (2012). Sharada and Shankar (2015) dealt with Soret, Dufour, thermal radiation and chemical reaction effects on MHD mixed convection flow of a Casson fluid over an exponentially stretching sheet, while Ahmad et al. (2016) discussed mixed convection of MHD Jeffry fluid flow over an exponentially stretching sheet. However the studies by Sharada and Shankar (2015) and Ahmad et al. (2016) were limited to prescribed surface temperature. To the best knowledge of the authors', mixed convection of Casson fluid over an exponentially stretching sheet with Newtonian heating has never been considered before. Hence, this study is aim to fill in the gap.

Problem Formulation

Consider a mixed convection boundary layer flow of Casson fluid over a vertical flat sheet. The sheet is heated due to Newtonian heating and flow is generated by stretching the sheet of velocity $u_w(x) = U_o e^{x/L}$, where U_o is the reference velocity and L is the reference length. The ambient temperature of Casson fluid is assumed to be constant at T_∞. Let x- and y-axis be the axes along the continuous sheet and perpendicular to it, respectively. Invoking Boussinesq and boundary layer approximations, the motion of the flow are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \upsilon\left(1 + \frac{1}{\beta}\right)\frac{\partial^2 u}{\partial y^2} + g\beta_T(T - T_\infty), \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}, \tag{3}$$

along with boundary conditions

$$u = u_w(x),\quad v = 0,\quad \frac{\partial T}{\partial y} = -h_s T \quad \text{at} \quad y = 0,$$
$$u \to 0,\quad T \to T_\infty \quad \text{as} \quad y \to \infty, \tag{4}$$

where u and v are the velocity along x- and y-directions, respectively. υ is the kinematic viscosity of the fluid, β is the Casson fluid parameter, g is the gravitational acceleration, β_T is the thermal expansion coefficient, T is the fluid temperature within the boundary layer, α is the thermal diffusivity of the fluid and h_s is the convective heat transfer coefficient. For the sake of similarity solution, the heat transfer parameter for Newtonian heating is assumed as $h_s = h_0 e^{x/2L}$, where h_0 is constant.

II. Methods

The governing Eqs. (1)-(3), subject to the boundary condition (4) can be expressed in a simpler form by introducing the following transformation (El Aziz 2009, Sharada and Shankar 2015):

$$u = U_o e^{x/L} f'(\eta), \quad v = -\sqrt{\frac{\upsilon U_o}{2L}}\, e^{x/2L} \left[f(\eta) + \eta f'(\eta) \right],$$
$$\eta = \sqrt{\frac{U_o}{2\upsilon L}}\, e^{x/2L} y, \quad \theta(\eta) = \frac{T - T_\infty}{T_\infty}, \tag{5}$$

where η is the similarity variable, $f\ (\eta)$ and $\theta\ (\eta)$ are the dimensionless stream function and dimensionless temperature, respectively, and prime denotes differentiation with respect to η. From (5), we get

$$\frac{\partial u}{\partial x} = \frac{U_o e^{x/L}}{L} f'(\eta) \ + \ \frac{\eta}{2} \frac{U_o e^{x/L}}{L} f''(\eta)$$
$$\frac{\partial u}{\partial y} = U_o e^{3x/2L} \sqrt{\frac{U_o}{2\upsilon L}}\ f''(\eta)$$
$$\frac{\partial^2 u}{\partial y^2} = U_o{}^2 \frac{e^{2x/L}}{2\upsilon L} f'''(\eta)$$
$$\frac{\partial v}{\partial y} = -\frac{U_o e^{x/L}}{L} f'(\eta) \ - \ \frac{\eta}{2} \frac{U_o e^{x/L}}{L} f''(\eta) \tag{6}$$
$$\frac{\partial T}{\partial x} = \frac{\eta}{2L} T_\infty \theta'(\eta)$$
$$\frac{\partial T}{\partial y} = e^{x/2L} \sqrt{\frac{U_o}{2\upsilon L}}\ T_\infty\ \theta'(\eta)$$
$$\frac{\partial^2 T}{\partial y^2} = e^{x/L} \frac{U_o}{2\upsilon L}\ T_\infty\ \theta''(\eta)$$

Putting (6) into Eq. (1) – (4), Eq. (1) is automatically satisfied and Eqs. (2) - (4) reduced to

$$\left(1 + \frac{1}{\beta}\right) f''' + f f'' - 2f'^2 + 2\lambda\theta = 0, \tag{7}$$
$$\theta'' + Pr\, f\, \theta' = 0, \tag{8}$$

with boundary conditions

$$f(\eta) = 0, \quad f'(\eta) = 1, \quad \theta'(\eta) = -Bi\left[1 + \theta(0)\right] \ \text{ at } \ \eta = 0,$$
$$f'(\eta) \to 0, \quad \theta(\eta) \to 0 \quad \text{ as } \ \eta \to \infty, \tag{9}$$

where prime denotes differentiation with respect to η, $\lambda = \frac{Gr}{Re^2}$ is the mixed convection parameter with $Gr = \frac{g \beta_T T_\infty L^3}{\nu^2}$ and $Re = \frac{U_o L}{\nu}$ are Grashoff number and Reynolds number, respectively. $Pr = \frac{\upsilon}{\alpha}$ is the Prandtl number and $Bi = h_o \sqrt{\frac{2\upsilon L}{U_o}}$ is the conjugate parameter for the Newtonian heating. It should be pointed out that the flow is dominated by forced convection if $\lambda = 0$ whilst $\lambda > 0$ and $\lambda < 0$ corresponds to assisting and opposing flow, respectively.

Numerical Methods

Eqs. (7) - (8) subject to (9) were solved using Keller box method for some arbitrary values of non-Newtonian fluid parameter β, mixed convection parameter λ and Newtonian heating parameter given by Biot number *Bi.* There are four steps required using this method as detailed out by Cebeci and Bradshaw (1988), i.e:

(i) Reduce (7) and (8) to first-order system.

$$\begin{aligned} f' &= u, \\ u' &= v, \\ \theta' &= s, \\ \left(1+\frac{1}{\beta}\right)v' + f\,v - 2u^2 + 2\lambda\theta &= 0, \\ s' + Pr\, f\, s &= 0, \end{aligned} \tag{10}$$

(ii) Write the difference equations using central differences for (10)

$$\begin{aligned} \frac{f_j - f_{j-1}}{h_j} &= \frac{u_j + u_{j-1}}{2} = u_{j-1/2}, \\ \frac{u_j - u_{j-1}}{h_j} &= \frac{v_j + v_{j-1}}{2} = v_{j-1/2}, \\ \frac{\theta_j - \theta_{j-1}}{h_j} &= \frac{s_j + s_{j-1}}{2} = s_{j-1/2}, \\ \left(1+\frac{1}{\beta}\right)\left(\frac{v_j - v_{j-1}}{h_j}\right) + (f\,v)_{j-1/2} - 2\left(u_{j-1/2}\right)^2 + 2\lambda\theta_{j-1/2} &= 0, \\ \left(\frac{s_j - s_{j-1}}{h_j}\right) + Pr(f\,s)_{j-1/2} &= 0, \end{aligned} \tag{11}$$

Rewriting (11),

$$\begin{aligned} f_j - f_{j-1} - \frac{h_j}{2}\left(u_j + u_{j-1}\right) &= 0, \\ u_j - u_{j-1} - \frac{h_j}{2}\left(v_j + v_{j-1}\right) &= 0, \\ \theta_j - \theta_{j-1} - \frac{h_j}{2}\left(s_j + s_{j-1}\right) &= 0, \end{aligned} \tag{12}$$

$$\left(1+\frac{1}{\beta}\right)\left(\frac{v_j - v_{j-1}}{h_j}\right)+\frac{1}{4}\left(f_j + f_{j-1}\right)\left(v_j + v_{j-1}\right)$$
$$-\frac{1}{2}\left(u_j + u_{j-1}\right)^2+\lambda\left(\theta_j + \theta_{j-1}\right)=0,$$
$$\left(\frac{s_j - s_{j-1}}{h_j}\right)+\frac{Pr}{4}\left(f_j + f_{j-1}\right)\left(s_j + s_{j-1}\right)=0,$$

(iii) Linearize (12) using Newton's method by introducing the following iterations

$$\begin{aligned} f_j^{(k+1)} &= f_j + \delta f_j^{(k)}, \\ u_j^{(k+1)} &= u_j + \delta u_j^{(k)}, \\ v_j^{(k+1)} &= v_j + \delta v_j^{(k)}, \\ \theta_j^{(k+1)} &= \theta_j + \delta \theta_j^{(k)}, \\ s_j^{(k+1)} &= s_j + \delta s_j^{(k)}, \end{aligned} \tag{13}$$

Substituting (13) into (12) and dropping superscript (k) and the quadratic terms of, $\delta f_j^{(k)}, \delta u_j^{(k)}, \delta v_j^{(k)}, \delta \theta_j^{(k)}$
and $\delta s_j^{(k)}$ we get

$$\begin{aligned} &\left(\delta f_j + \delta f_{j-1}\right) - \frac{h_j}{2}\left(\delta u_j + \delta u_{j-1}\right) = (r_1)_{j-1/2}, \ \left(\delta u_j + \delta u_{j-1}\right) - \frac{h_j}{2}\left(\delta v_j + \delta v_{j-1}\right) = (r_2)_{j-1/2}, \\ &\left(\delta \theta_j + \delta \theta_{j-1}\right) - \frac{h_j}{2}\left(\delta s_j + \delta s_{j-1}\right) = (r_3)_{j-1/2}, \\ &(a_1)\delta v_j + (a_2)\delta v_{j-1} + (a_3)\delta f_j + (a_4)\delta f_{j-1} \\ &\quad + (a_5)\delta u_j + (a_6)\delta u_{j-1} + (a_7)\delta \theta_j + (a_8)\delta \theta_{j-1} = (r_4)_{j-1/2}, \\ &(b_1)\delta s_j + (b_2)\delta s_{j-1} + (b_3)\delta f_j + (b_4)\delta f_{j-1} = (r_5)_{j-1/2}, \end{aligned} \tag{14}$$

where

$$\begin{aligned} (r_1)_{j-1/2} &= \left(f_{j-1} - f_j\right) + h_j u_{j-1/2}, \\ (r_2)_{j-1/2} &= \left(u_{j-1} - u_j\right) + h_j v_{j-1/2}, \\ (r_3)_{j-1/2} &= \left(\theta_{j-1} - \theta_j\right) + h_j s_{j-1/2}, \\ (r_4)_{j-1/2} &= \left(1+\frac{1}{\beta}\right)\left(v_{j-1} - v_j\right) - h_j f_{j-1/2} v_{j-1/2} \\ &\quad + 2h_j u_{j-1/2}^2 - 2\lambda h_j \theta_{j-1/2}, \\ (r_5)_{j-1/2} &= \left(s_{j-1} - s_j\right) - Pr h_j f_{j-1/2} s_{j-1/2}, \end{aligned}$$

$$(a_1)_j = \left(1+\frac{1}{\beta}\right)+\frac{h_j}{2} f_{j-1/2},$$

$$(a_2)_j = -\left(1+\frac{1}{\beta}\right)+\frac{h_j}{2} f_{j-1/2},$$

$$(a_3)_j = (a_4)_j = \frac{h_j}{2} v_{j-1/2},$$

$$(a_5)_j = (a_6)_j = -2h_j u_{j-1/2},$$

$$(a_7)_j = (a_8)_j = \lambda h_j,$$

$$(b_1)_j = 1+\frac{h_j}{2} Pr f_{j-1/2},$$

$$(b_2)_j = -1+\frac{h_j}{2} Pr f_{j-1/2},$$

$$(b_3)_j = (b_4)_j = \frac{h_j}{2} Pr s_{j-1/2},$$

(iv) Solve Eq. (14) by block-tridiagonal elimination technique. Three cases were considered; i.e $j = 1$, $j = J - 1$ and $j = J$ which later can be written in vector-matrix form as

$$\boldsymbol{A\delta = r} \qquad (15)$$

where

$$A = \begin{bmatrix} [A_1] & [C_1] & & & & \\ [B_2] & [A_2] & [C_2] & & & \\ & & \ddots & & & \\ & & \ddots & & & \\ & & \ddots & & & \\ & & & [B_{J-1}] & [A_{J-1}] & [C_{J-1}] \\ & & & & [B_J] & [A_J] \end{bmatrix},$$

$$\delta = \begin{bmatrix} [\delta_1] \\ [\delta_2] \\ \vdots \\ [\delta_{J-1}] \\ [\delta_J] \end{bmatrix}, \quad r = \begin{bmatrix} [r_1] \\ [r_2] \\ \vdots \\ [r_{J-1}] \\ [r_J] \end{bmatrix}$$

with

$$A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ \frac{-h_1}{2} & 0 & 0 & \frac{-h_1}{2} & 0 \\ 0 & -1 & 0 & 0 & \frac{-h_1}{2} \\ (a_2)_1 & (a_8)_1 & (a_3)_1 & (a_1)_1 & 0 \\ 0 & 0 & (b_3)_1 & 0 & (b_1)_1 \end{bmatrix},$$

$$A_j = \begin{bmatrix} \frac{-h_j}{2} & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & \frac{-h_j}{2} & 0 \\ 0 & -1 & 0 & 0 & \frac{-h_j}{2} \\ (a_6)_j & (a_8)_j & (a_3)_j & (a_1)_j & 0 \\ & & (b_3)_j & & (b_1)_j \end{bmatrix},$$

$$B_j = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & \frac{-h_j}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{-h_j}{2} \\ 0 & 0 & (a_4)_j & (a_2)_j & 0 \\ 0 & 0 & (b_4)_j & 0 & (b_2)_j \end{bmatrix},$$

$$C_j = \begin{bmatrix} \frac{-h_j}{2} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ (a_5)_j & (a_7)_j & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad 1 \le j \le J-1,$$

$$[\delta_1] = \begin{bmatrix} \delta v_0 \\ \delta \theta_0 \\ \delta f_1 \\ \delta v_1 \\ \delta s_1 \end{bmatrix}, \quad [\delta_j] = \begin{bmatrix} \delta u_{j-1} \\ \delta \theta_{j-1} \\ \delta f_j \\ \delta v_j \\ \delta s_j \end{bmatrix}, \quad 2 \le j \le J,$$

and

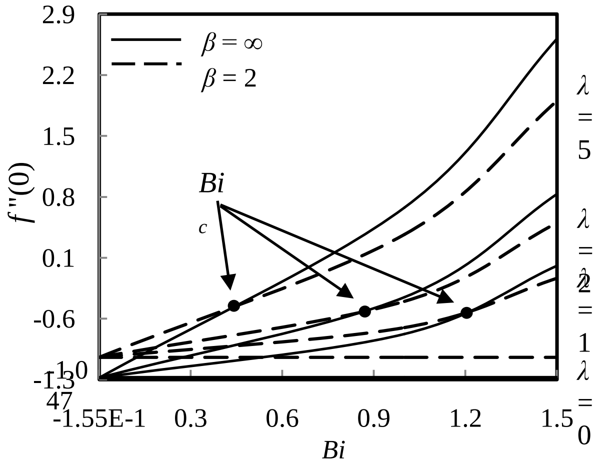
$$[r_j] = \begin{bmatrix} (r_1)_{j-1/2} \\ (r_2)_{j-1/2} \\ (r_3)_{j-1/2} \\ (r_4)_{j-1/2} \\ (r_5)_{j-1/2} \end{bmatrix}.$$

Eq. (15) was solved using block elimination method. The problem was coded in Matlab software and two quantities of interest; i.e $f''(0)$ and $\theta(0)$ were generated from the program.

III. Results and Discussion

The effect of Newtonian heating which is given by the parameter *Bi* is seen to give outstanding impact towards $f''(0)$ and $\theta(0)$ as depicted in Fig. 1(a) and (b), respectively. The increment of *Bi* number increases both $f''(0)$ and $\theta(0)$ except when the flow is dominated by forced convection (λ = 0). This is expected due to the absence of λ in Eq. 7 which reflects the independency of fluid flow towards the thermal flow and causing fixed $f''(0)$. Furthermore, as $\lambda > 0$ and *Bi* increases, $f''(0)$ is getting higher. However, this scenario does not hold for $\theta(0)$. The impact of mixed convection parameter λ on $\theta(0)$ is very much profound for *Bi* >> 0.9 and no remarkable change for *Bi* << 0.9. Arbitrary values of λ in the absence of *Bi* number (*Bi* = 0) causing the $f''(0)$ remains fixed at certain value of $f''(0)$, and the surface temperature $\theta(0)$= 0 due to no heating at the sheet. The effect of non-Newtonian fluid (Casson fluid) given by β is found to give interesting behaviors to the $f''(0)$ and $\theta(0)$. At fixed λ and $Bi < Bi_c$, $f''(0)$ is higher for Casson fluid (β = 2) as compared to Newtonian fluid ($\beta = \infty$) and opposite phenomenon occur beyond this value; i.e $Bi > Bi_c$. It should be stressed out that Bi_c decreases as λ increases. This condition does not hold for $\theta(0)$. The determination of $\theta(0)$ is very much depends on λ and β. For λ = 1, there is no effect of β onto $\theta(0)$ is seen. However for λ >> 1, $\theta(0)$ is found to be higher when the flow is dominated by Casson fluid (β = 2) which contradict the behavior of $\theta(0)$ for λ << 1.

Figs. 2-3 are depicted to further visualize the effect of mixed convection parameter λ onto $f''(0)$ and $\theta(0)$, respectively, for several values of *Bi* number and for the case of Newtonian fluid ($\beta = \infty$) and Casson fluid (β = 2). Here, we employ both values of λ, i.e $\lambda \geq 0$ and $\lambda < 0$. For λ = 0, the fluid flow is no longer influenced by *Bi* number which results of fixed $f''(0)$.



(b)

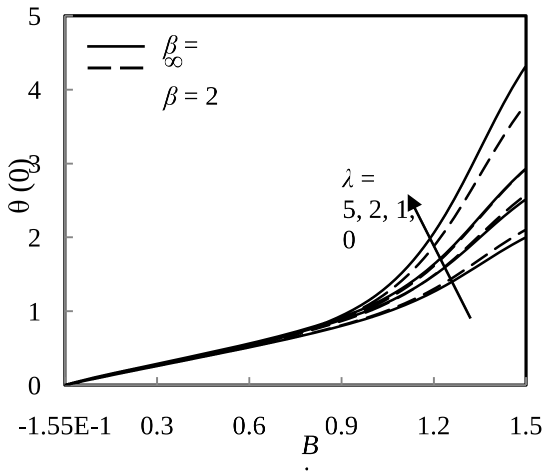


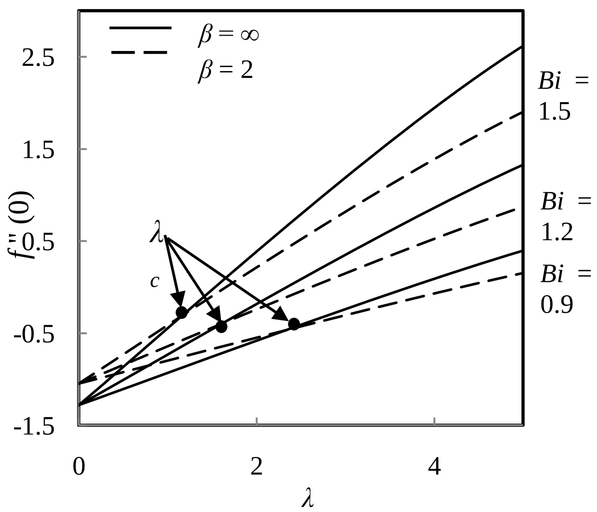
Fig. 1. **(a)** $f''(0)$ and **(b)** $\theta(0)$ for various values of *Bi* and $\lambda \geq 0$

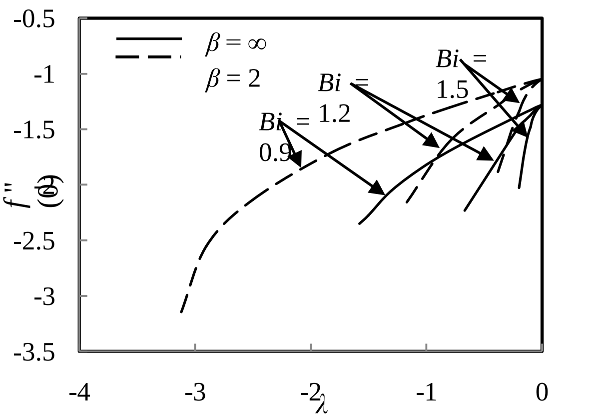
For $\lambda >> 0$ (assisting flow), the increment of λ is found to increase $f''(0)$ and decrease $\theta(0)$ as potrayed in Figs. 2a and 3a. This is in line with the results obtained in Fig. 1. However, the effect of λ on $\theta(0)$ is very much profound for higher value of *Bi* number (*Bi* = 1.5) in comparison to smaller values of *Bi* number; i.e *Bi* = 1.2 and *Bi* = 0.9 due to the abrupt decrement of $\theta(0)$. This is projected as mixed convection parameter λ very much affecting the fluid flow as compared to the thermal flow. The presence of *Bi* number changes the thermal flow due to the heating at the surface. The effects of Newtonian and Casson fluid also can be seen from these figures. The property of the fluid is found to give different characteristics towards the $f''(0)$ and $\theta(0)$ at certain value of λ_c.

Figs. 2b and 3b depicted the behavior of $f''(0)$ and $\theta(0)$ for $\lambda < 0$, respectively. Here, contrary scenarios occurred as for the $f''(0)$ and $\theta(0)$ when $\lambda \geq 0$ (figs. 2a and

3a). It should be pointed out that both $f''(0)$ and $\theta(0)$ are limited to certain negative values of λ subject to the *Bi* number and β under consideration. For instance, the smaller the *Bi* number, the wider the range of λ can be computed. Also, it is worth mentioning that this value can be improved by imposing Casson fluid (β = 2) instead of non-Newtonian fluid ($\beta = \infty$).

(a)





(b)

Fig. 2. Influence of *Bi* number onto $f''(0)$ when **(a)** $\boldsymbol{\lambda > 0}$ and **(b)** $\boldsymbol{\lambda < 0}$

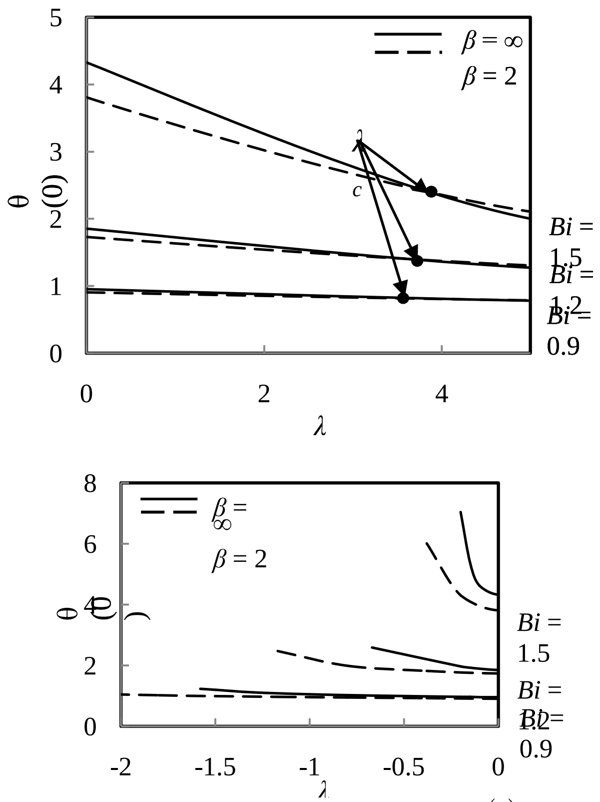


Fig. 3. Influence of *Bi* number onto $\theta(0)$ when **(a) λ > 0** and **(b) λ < 0**

IV. Conclusions

Numerical solution for the mixed convection Casson fluid over an exponentially stretching sheet with Newtonian heating is considered. Various arbitrary values of the mixed convection λ, *Bi* number for both Newtonian and Casson fluid towards $f''(0)$ and $\theta(0)$ are depicted in form of graphs. The increment of λ ($\lambda >> 0$) contributes to increment of $f''(0)$ and decrement of $\theta(0)$ (only at high value of *Bi* number). Limited solutions were obtained for $\lambda << 0$. *Bi* number is expected to increase both $f''(0)$ and $\theta(0)$ except for $\lambda = 0$.

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