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Perspective

The Miracle in the Iron and the Ising Model of the Ferromagnet

Nasir Ganikhodjaev*
Department of Computational and Theoretical Sciences, Faculty of Science,
International Islamic University Malaysia, 25200 Kuantan, Malaysia.

Abstract
It is well known that the Qur'an has many miraculous attributes proving that it is revelation from Allāh. In this paper we consider one of them, namely, the miracle in the Iron. One of the main and very useful property of the Iron is to be magnet. In 1925 Ising wrote his doctoral thesis on a model now called the Ising model. He tried to explain, using this model, certain empirically observed facts about ferromagnetic materials. This paper is an attempt to present the basic ideas of the Ising model and its application to a wider audience.

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Keywords: Iron, spontaneous magnetization, Ising model.

Abstrak
Al-Qur'an sememangnya diketahui umum mempunyai ciri-ciri keajaiban sebagai bukti yang menunjukkan bahawa kitab tersebut diturunkan oleh Allah. Dalam kertas ini, kami mengambil kira satu daripada bukti tersebut, iaitu keistimewaan besi. Satu daripada ciri besi yang utama dan yang paling berguna ialah magnet. Dalam tahun 1952, Ising telah menulis tesis kedoktorannya berasaskan model yang kini dikenali sebagai model Ising. Dengan menggunakan model ini, beliau cuba menerangkan fakta-fakta empirikal tertentu yang berkaitan dengan bahan-bahan ferromagnet. Kertas ini cuba menjelaskan idea-idea asas tentang model Ising dan aplikasinya kepada pembaca umum.

Kata kunci: Besi, pemagnetan secara spontan, model Ising

"Verily this (the Qur'an) is a Revelation from the Lord of the Worlds"

(The Qur'an 26:192)

Introduction

Fourteen centuries ago, Allāh sent down the Qur'an to mankind as a book of guidance. He called upon people to be guided to the truth by adhering to this book. From the day of its revelation to the day of judgement, this last divine book will remain the sole guide for humanity. The matchless style of the Qur'an and the superior wisdom in it are definite evidence that it is the word of Allāh. In addition, the Qur'an has many miraculous attributes proving that it is a revelation from Allāh. One of these attributes is the fact that a number of scientific truths that we have only been able to uncover by the technology of the 21st century were stated in the Qur'an 1,400 years ago. Of course the Qur'an is not a book of science. However, many scientific facts that are expressed in an extremely concise and profound manner in its verses have only been discovered with the

**Corresponding author: Nasir Ganikhodjaev,*
Professor, Department of Computational and Theoretical Sciences, Faculty of Science, International Islamic University Malaysia, 25200 Kuantan, Pahang, Malaysia.
Phone: (609) 571 6752 (O); Fax: (609)571 6781

technology of the 21st century. These facts could not have been known at the time of the Qur'an's revelation, and this is still more proof that the Qur'an is the word of Allāh. In order to understand the scientific miracle of the Qur'an, we must first take a look at the level of science at the time when this holy book was revealed. In the 7th century, when the Qur'an was revealed, Arab society had many superstitious and groundless beliefs where scientific issues were concerned. Lacking the technology to examine the universe and nature, these early Arabs believed in legends inherited from past generations. They supposed, for example, that mountains supported the sky above. They believed that the earth was flat and that there were high mountains at its both ends. It was thought that these mountains were pillars that kept the vault of heaven high above. However all these superstitious beliefs of Arab society were eliminated with the Qur'an. In *Sura al-Ra'd*, verse 2, it was said:

"Allah is He Who raised the heavens without any pillars that ye can see; is firmly established on the throne (of authority); He has subjected the sun and the moon (to his Law)! Each one runs (its course) for a term appointed. He doth regulate all affairs, explaining the signs in detail, that ye may believe with certainty in the meeting with your Lord"
(The Qur'an 13:2).

This verse invalidated the belief that the sky remains above because of the mountains. In many other subjects, important facts were revealed at a time when no one could have known them. The Qur'an, which was revealed at a time when people knew very little about astronomy, physics, or biology, contains key facts on a variety of subjects such as the creation of the universe, the creation of the human being, the structure of the atmosphere, and the delicate balances that make life on earth possible (Yahya 2001).

The miracle in the Iron

Iron is one of the elements highlighted in the Qur'an. In *Sura al-Hadid* (Iron), we are informed:

"We sent aforetime our apostles with Clear Signs and sent down with them the Book and the Balance (of Right and Wrong), that men may stand forth in justice; and We sent down Iron, in which is (material for) mighty war, as well as many benefits for mankind, that Allah may test who it is that will help, Unseen, Him and His apostles: For Allah is Full of Strength, Exalted in Might (and able to enforce His Will)"
(The Qur'an 57:25)

The word "sent down," particularly used for iron in the verse, could be thought of having a metaphorical meaning to explain that iron has been given to benefit people. But when we take into consideration the literal meaning of the word, which is, "being physically sent down from the sky", we realize that this verse implies a very significant scientific miracle. This is because modern astronomical findings have disclosed that the iron found in our world has come from the giant stars in outer space. The heavy metals in the universe are produced in the nuclei of big stars. Our solar system, however, does not possess a suitable structure for producing iron on its own. Iron can only be produced in much bigger stars than the Sun, where the temperature reaches a few hundred million degrees. When the amount of iron exceeds a certain level in a star, the star can no longer accommodate it, and eventually it explodes in what is called a "nova" or a "supernova". As a result of this explosion, meteors containing iron are scattered around the universe, and they move through the void until attracted by the gravitational force of a celestial body. All this shows that iron did not form on the Earth, but was carried from exploding stars in space via meteors, and was "sent down to earth", in exactly the same way as stated in the verse: It is clear that this fact could not have been scientifically known in the 7th century, when the Qur'an was revealed (Yahya 2001)

Magnets and its Applications

One of the main and very useful property of the Iron is to be magnet. In search of more efficient transportation techniques, power and energy generation, magnetic technology is gaining considerable significance in today's world. Along with the invention of fire and wheels, the discovery of magnet is considered one of the greatest achievements.

Available in various shapes and sizes, for home as well as industrial uses, magnet is a material that produces an invisible field named as magnetic field, which is used to attract iron objects. The two primary varieties of magnets are the permanent magnets and the electromagnets. Permanent magnets are created from a magnetized material, having a persistent magnetic field. The electromagnet on the other hand, is one which consists of a coil wire, which behaves as a magnet when an electric current is passed through it.

Nowadays, magnets are used in a multitude of application areas. The modern world is so much dependent on magnets that its flow might come to a halt in their absence. There is nothing wrong in saying that magnets are almost indispensable for the survival of the present world.

Different Applications of Magnets

In industries, magnets are used in various equipment like pulleys, separators, sweepers and welding devices, to name a few. They are the driving force behind electric motors and generators. They are used in audio cassettes, floppy disks and hard disks, the entire data is recorded on a thin coating film. Another usage is in health sector where hospitals are using magnetic imaging resonance techniques to scan the organs and for various surgical purposes. The use of magnets is also common in magnetic therapy devices like magnetic waist belts, magnetic mattresses, magnetic massager, knee magnets and so. The entire concept of Maglev train is based on the repulsion between electromagnets. Due to the repulsion only, the train floats above the track at a drastic speed. Certain equipment, which are common in households these days, also make use of magnets. These equipment include headphones, speakers, refrigerators, television, water pumps, radio, telephone, etc. Nowadays, magnets are also used for making certain special kind of jewelry items like necklaces, bracelets, earrings and pendants. The commonly used credit, debit and ATM cards these days are based on the magnetic technology. They possess a magnetic strip which contains all the necessary information. Other miscellaneous uses of magnets include use in compasses, vinyl magnet sheets, science projects, toys, tools, fail-safe devices, and so on.

Subject of Statistical Mechanics

Science is the human endeavor to understand the nature of the world into which we are born. This quest for understanding is driven by practical needs as well as by an innate curiosity which, whatever its evolutionary origin and utility, goes far beyond the utilitarian. Even very young children have this basic urge to explore and examine. When maintained into adulthood it gives rise to all human creativity including that in the sciences, both theoretical and experimental. Main goal of the statistical physics consists of in finding out how the dynamics of the microscopic components of matter, such as atoms and molecules, determine the behavior of macroscopic objects containing very many atoms, objects that we can see and touch, like a glass of water or a piece of metal (Lebowitz 2004). This is subject of statistical mechanics which provides a mathematical framework for describing how well-organized higher level structures or behavior may result from the random, non-directed activity of a very large number of interacting lower level entities. Fortunately, an understanding of many aspects of the behavior of macroscopic systems- such as the boiling and freezing of water- can be obtained from simplified models of the structure and interaction of atoms. We can often take as our starting point Feynman's description of atom as " little particles that move around in perpetual motion, attracting each other when they are a little distance apart, but repelling upon being squeezed into one another". The degree to which this simple picture gives predictions which are not only qualitatively correct but in many cases also highly accurate, is remarkable, since the structure of real atoms is governed by quantum mechanics and is much more complicated than Feynman's rather crude classical picture. Statistical mechanics explains how macroscopic phenomena originate in the cooperative behavior of these " little particles". Some of this phenomena are simply the effects of the combined actions of many individual atoms; for instance, the pressure exerted by a gas on the walls of its container is due to the continual bombardment of the walls by very many gas molecules. But other phenomena are examples of emergent behavior, they have no direct counterpart in the properties or dynamics of individual atoms. Particularly, fascinating and important example of such emergent phenomena is phase transition, such as occur in the boiling and freezing of water. Here dramatic changes in structure or pressure without any change in the structure of the individual atoms or molecules making up the system. For example the volume occupied by a kilogram of water molecules at atmospheric pressure , while changing only very little when the temperature increase is between 5° C and 95° C, increases by a very large factor when the temperature changes from 99.9999° C to 100.0001° C. Even more dramatic things happen in the freezing transition around 0° C where there are essentially "infinite" changes in some property, like fluidity.

The Ising Model

In recent years, a new type of stochastic process, called a Markov random fields, has been introduced in the theory of probability. The motivation for looking at such processes came originality from statistical physics, but it is clear that these processes form a natural generalization of Markov processes in which a time index is replaced by a space index.

The concept of a Markov random field came from attempts to put into a general probabilistic setting a very specific model named after the German physicist Ernst Ising. Ising was a student of Lenz and wrote his doctoral thesis on a model now called the Ising model. He tried to explain, using this model, certain empirically observed facts about ferromagnetic materials. When Ising published a summary of his results, he stated that the model was suggested by Lenz. For interesting historical discussion of the origins and development of the Ising model see Brush (1967).

One-dimensional Ising model on finite set.

The first formulation given by Ising is as follows: consider a sequence , $\Lambda_n = \{0,1,2;\cdot\cdot,n\}$ of points on the line. At each point, or site, there is a small dipole or "spin" which at any given moment is in one of two positions, "up" or "down". It is customary now to indicate the spins in the form of a configuration as shown in Figure 1.

Figure 1: A configuration on Λ_n.

Following Ising, we are going to put a probability measure on the set of all possible configurations. Such a measure is called a *random field* . Using current probability notation we choose as sample space the space Ω of all sequences

$$\omega = (\omega_0, \omega_1, \cdots, \omega_n)$$

where $\omega_j = +$ or $-$ with"+" indicating a spin up and "-" a spin down.

Define the spin σ_j as a function $\sigma_j : \Omega \to \{-1,1\}$ such that $\sigma_j(\omega) = 1$ if $\omega_j = +$ and $\sigma_j(\omega) = -1$ if $\omega_j = -$.

Ising defined a probability measure on Ω as follows. To each configuration ω an *energy* $U(\omega)$ is assigned by

$$U(\omega) = -J\sum_{i,j}\sigma_i(\omega)\sigma_j(\omega) - mH\sum_i \sigma_i(\omega).(1)$$

Here the first sum is taken over all pairs i, j of points which are one unit apart. (We count each pair only once.)

The first sum represents the energy caused by interaction of the spins. Ising made the simplifying assumption that only interaction between neighboring spins need be taken into account. The constant J is a property of the material being considered. The case $J > 0$ is called the *attractive case*. The reason for this is that the interaction tends to keep neighboring spins aligned the same. The case $J<0$ is called the *repulsive case* since it tends to reinforce pairs in which the spins are of opposite orientation. The second term represents the effect of an external magnetic field of intensity H. The constant $m>0$ is a property of the material. In the attractive case, the first term contributes minimum energy when all of the spins are lined up in the same direction. The second term contributes minimum energy when all the spins are in the same direction as the external field.

Ising then assigned probabilities to configurations ω proportional to

$$e^{-\frac{1}{kT}U(\omega)} \tag{2}$$

where T is the absolute temperature (Kelvin scale)and k is a universal constant. The probability measure on Ω is thus given by

$$P(\omega)=\frac{e^{-\frac{1}{kT}U(\omega)}}{Z}, \tag{3}$$

where the normalizing constant Z , is defined by

$$Z=\sum_{\omega} e^{-\frac{1}{kT}U(\omega)}, \tag{4}$$

and is called the *partition function*.

Ising model on countable set Z_+ of nonnegative integers.

Now consider an infinite sequence , $S=\{0,1,2,\cdots\}$ of points on the line,i.e., the set of nonnegative integers. It is a countable set. Again at each point, or site, there is a small dipole or "spin" which at any given moment is in one of two positions, "up" or "down". Now the set of all possible configurations on S is the infinite uncountable set.

Again we choose as sample space the space Ω of all sequences

$$\omega=(\omega_0,\omega_1,\cdots,\omega_n,\cdots),$$

where $\omega_j=+$ or $-$ with"+" indicating a spin up and "-" a spin down.

Define the spin σ_j as a function $\sigma_j:\Omega\to\{-1,1\}$ such that $\sigma_j(\omega)=1$ if $\omega_j=+$ and $\sigma_j(\omega)=-1$ if $\omega_j=-$.

How can we define a probability measure on Ω? Again to each configuration ω an *energy* $U(\omega)$ is assigned by

$$U(\omega)=-J\sum_{i,j}\sigma_i(\omega)\sigma_j(\omega)-mH\sum_i\sigma_i(\omega). \tag{5}$$

Here the first sum is taken over all pairs i,j of points which are one unit apart. (We count each pair only once.). But now both sums are infinite sum, that is, they are the series.

It is easy to see that the series (5) diverge for any configuration ω, i.e., Hamiltonian (5) is formal, since we cannot operate with it.

For finite case Ising assigned probabilities to configurations ω as

$$P(\omega)=\frac{e^{-\frac{1}{kT}U(\omega)}}{Z}, \tag{6}$$

where the normalizing constant Z , is defined by

$$Z=\sum_{\omega} e^{-\frac{1}{kT}U(\omega)}. \tag{7}$$

Let $\Lambda_n=\{0,1;\cdot\cdot,n\}$ and Ω_n be a set of all configurations on Λ_n. Then $\Lambda_1\subset\Lambda_2\subset\cdots\subset\Lambda_n\subset\cdots$ with $S=\cup_{n=1}^{\infty}\Lambda_n$. Let us fix some n and let ω_n be a configuration on Λ_n. Now on infinite set $\Lambda_n^c=S\setminus\Lambda_n$ we will fix a configuration ω^n and call it a boundary condition.For any n with given boundary condition ω^n we have finitely many configurations on Λ_n. Now for given boundary condition ω^n energy of configuration ω_n is assigned by

$$U(\omega_n \mid \omega^n) = -J \sum_{<i,j>\in\Lambda_n} \sigma_i(\omega_n)\sigma_j(\omega_n) - J \sum_{i\in\Lambda_n, j\in\Lambda_n^c} \sigma_i(\omega_n)\sigma_j(\omega^n) - mH\sum_{i\in\Lambda_n}\sigma_i(\omega_n). \tag{8}$$

Here the first and second sum are taken over all pairs $<i,j>$ of nearest neighbour. Then (8) we can rewrite as

$$U(\omega_n \mid \omega^n) = -J\sum_{i=0}^{n-1}\sigma_i(\omega_n)\sigma_{i+1}(\omega_n) - J\sigma_n(\omega_n)\sigma_{n+1}(\omega^n) - mH\sum_{i=0}^{n}\sigma_i(\omega_n). \tag{9}$$

Now this energy is a finte number for any ω_n. We define the *conditional Gibbs state on* Λ_n *with boundary condition* ω^n *and Hamiltonian* (9) to be the state $\mu_n(\cdot \mid \omega^n)$ given by

$$\mu_n(\omega_n \mid \omega^n) = Z_n^{-1}(\omega^n)\exp(-\frac{1}{kT}U(\omega_n \mid \omega^n)), \tag{10}$$

for any $\omega_n \in \Omega_n$, where

$$Z_n(\omega^n) = \sum_{\omega_n\in\Omega_n} \exp(-\frac{1}{kT}U(\omega_n \mid \omega^n)) \tag{11}$$

Thus we have a sequence of conditional Gibbs states $\{\mu_n(\cdot \mid \omega^n)\}$. Now we will define a limit Gibbs state on Ω by following way.

Definition 1

We will say that μ is a limit Gibbs state on Ω, if for any n and for arbitrary boundary configuration ω^n the conditional probability with respect to μ given that the configuration $\omega \in \Omega$ is ω^n on Λ_n^c is the same as the conditional Gibbs state on Ω_n given by (10).

The main problem of equilibrium statistical physics is to describe all limit Gibbs states of given Hamiltonian.

When all boundary points ω^n are fixed as $+$ we have the *positive boundary* and when they are fixed as $-$ we have the *negative boundary*.

We begin by considering this question for the Ising model on Z_+: will the positive and negative boundaries ω^n give different probabilities for event $\{\omega \in \Omega : \omega(0) = +\}$ as n tends to infinity?

Assume $\Omega_n^+ = \{\omega_n \in \Omega_n : \omega_n(0) = +\}$ and $\Omega_n^- = \{\omega_n \in \Omega_n : \omega_n(0) = -\}$. Then $\Omega_n = \Omega_n^+ \cup \Omega_n^-$ and $\cup_n \Omega_n^+ = \{\omega \in \Omega : \omega(0) = +\}$, that is, Ω_n^+ tends to $\{\omega \in \Omega : \omega(0) = +\}$ as n tends to infinity.

For fixed n we divide the partition function $Z_n(\omega^n)$ into two sums

$$Z_n(\omega^n) = Z_n^+(\omega^n) + Z_n^-(\omega^n)$$

where

$$Z_n^+(\omega^n) = \sum_{\omega_n\in\Omega_n^+} \exp(-\frac{1}{kT}U(\omega_n \mid \omega^n))$$

and

$$Z_n^-(\omega^n) = \sum_{\omega_n\in\Omega_n^-} \exp(-\frac{1}{kT}U(\omega_n \mid \omega^n))$$

Thus $Z_n^-(\omega^n)$ sums over all configurations in Ω_n^- which assign $-$ to the origin site 0, and $Z_n^+(\omega^n)$ sums over all configurations in Ω_n^+ which assign $+$ to the origin site 0. We now compute the ratio of the probability of a $-$ at the origin to the probability of a $+$ at the origin,that is,

$$\frac{\mu_n(\Omega_n^- \mid \omega^n)}{\mu_n(\Omega_n^+ \mid \omega^n)}.$$

This ratio we can rewrite as

$$\frac{\mu_n(\sigma_0 = -1 \mid \omega^n)}{\mu_n(\sigma_0 = 1 \mid \omega^n)}$$

If we can find the limit of u_n as n tends to infinity, we will find the limiting ratio for the probability of a $-$ to the probability of a $+$ at the origin, that is,

$$\frac{\mu(\sigma_0 = -1)}{\mu(\sigma_0 = 1)} = \lim_{n\to\infty} \frac{\mu_n(\sigma_0 = -1 \mid \omega^n)}{\mu_n(\sigma_0 = 1 \mid \omega^n)}.$$

Assume $\beta = \frac{1}{kT}$, and $b = \exp(2\beta J); h = \exp(2\beta mH)$. Thus we can write

$$Z_n^+ = \exp(\beta J)\exp(\beta mH)Z_{n-1}^+ + \exp(-\beta J)\exp(\beta mH)Z_{n-1}^-.$$

Similarly

$$Z_n^- = \exp(-\beta J)\exp(-\beta mH)Z_{n-1}^+ + \exp(\beta J)\exp(-\beta mH)Z_{n-1}^-.$$

Thus

$$\frac{Z_n^-}{Z_n^+} = \frac{\exp(-\beta J)\exp(-\beta mH)Z_{n-1}^+ + \exp(\beta J)\exp(-\beta mH)Z_{n-1}^-}{\exp(\beta J)\exp(\beta mH)Z_{n-1}^+ + \exp(-\beta J)\exp(\beta mH)Z_{n-1}^-}$$

$$= \exp(-2\beta mH)\frac{Z_{n-1}^- + \exp(-2\beta J)Z_{n-1}^+}{\exp(-2\beta J)Z_{n-1}^- + Z_{n-1}^+}$$

Then we have

$$u_n = \frac{h^{-1}(u_{n-1}+b)}{bu_{n-1}+1} \qquad (12)$$

This ratio, of course, determines the probability of a $+$ at the origin site 0 since the sum of the two probabilities is 1. We define

$$u_n = \frac{\mu_n(\sigma_0 = -1 \mid \omega^n)}{\mu_n(\sigma_0 = 1 \mid \omega^n)} = \frac{Z_n^-}{Z_n^+}.$$

The value u_1 may be obtained by considering a graph Λ_1. With positive boundary

$$u_1 = \frac{P\begin{pmatrix} + \\ - \end{pmatrix}}{P\begin{pmatrix} + \\ + \end{pmatrix}} = (bh)^{-1}$$

Similarly, with negative boundary

$$u_1 = bh^{-1}$$

We can then compute u_n for $n > 1$ by using the fact that

$$u_n = f(u_{n-1})$$

where

$$f(x) = \frac{h^{-1}(x+b)}{bx+1}$$

We then ask if the sequence u_n a tends to limit u, and, if so, does this limit depend upon the

Figure 2: Distribution with two peaks

value of u_1 ? Passing to limit in (12) we have following equation

$$u = \frac{h^{-1}(u+b)}{bu+1} \qquad (13)$$

Evidently this equation has a single positive root, that is, for any boundary condition there exists single limit Gibbs measure.

What Ising hoped to establish

Basically, Ising was interested in the case of no Éexterior field,i.e. $H = 0$. It was thought that for sufficiently low temperatures, even if the spins were random to begin with, they would tend to move to a state of lower energy, i.e., mostly up or mostly down, forming a magnet

Thus, in equilibrium, if $n_+(\omega)$ is the number of up spins and $n_-(\omega)$ is the number of down spins, the total magnetization

$$M(\omega) = n_+(\omega) - n_-(\omega)$$

would be expected to have a distribution with two peaks as in figure 2. This would result in "spontaneous magnetization", i.e., the spins would tend to be either mostly + or mostly -. Ising remarked that this did not occur for one-dimensional case and then went on to consider the case of two dimensions. Here he made a mistake which held back the development of his model for many years. He argued that this magnetic effect would be even more noticeable if in two dimensions he allowed different attractive forces J_1 and J_2 for the two possible directions. He in fact considered the case where the force J_2 in the vertical direction went to infinity and remarked that then all rows would be forced to conform to each other as in figure 3.

+	-	+	-	+	+
+	-	+	-	+	+
+	-	+	-	+	+
+	-	+	-	+	+
+	-	+	-	+	+
+	-	+	-	+	+

Figure 3: Reducing to one-dimensional case.

He then could apply his one-dimensional result and was again led to a unimodal distribution for the magnetization M. Thus he came to the conclusion that his model was too crude to explain magnetization. Ising was forced to leave Germany in 1936 and was cut off from the scientific community and unable to pursue his work. About the only immediate attention given to his paper was by Heisenberg (1928) who used the apparant failure as a reason to introduce a more complicated model. However, interest in the Ising model was revived by Bethe (1935) and others interested in the applications as formation of binary alloys. Peierls (1936) developed a method to show that in two and more dimensions the " spontaneous magnetization effect" could be seen to occur in the Ising model. His proof was not quite rigorous and careful proofs were given later by Dobrushin (1968) and Griffiths (1964) independently. Note that this proof is rather complicated, however we can show the existing of spontaneous magnetization effect for Ising model on the Cayley tree (figure 4) using more simple approach.

Spontaneous Magnetization on a Cayley Tree

A Cayley tree Γ^k of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles with exactly $k+1$ edges issuing from each vertex. Let denote the Cayley tree as $\Gamma^k = (V, \Lambda)$, where V is the set of vertices of Γ^k, Λ is the set of edges of Γ^k. Two vertices x and y, $x, y \in V$ are called *nearest neighbors* if there exists an edge $l \in \Lambda$ connecting them, which is denoted by $l = <x, y>$. The distance $d(x, y), x, y \in V$, on the Cayley tree, is the number of edges in the shortest path from x to y. For a fixed $x^0 \in V$ we set $V_n = \{x \in V \mid d(x, x^0) \leq n\}$ and L_n denotes the set of edges in V_n. We shall call the set

$$W_n = \{x \in V : d(x, x^0) = n\}$$

the $n-$th level of the tree Γ^k and

$$V_n = \cup_{i=0}^{n} W_i$$

the "$n-$ storey house". It is evidently that $V_1 \subset V_2 \subset \cdots \subset V_n \subset \cdots$ with $V = \cup_{n=1}^{\infty} V_n$.

A configuration σ on V is defined as as function $\sigma : V \to \{-1,1\}$. Let Ω be a set of all configurations on V . It is evidently that Ω is an infinite set. The formal Hamiltonian on Ω is defined as

$$H(\sigma) = -J \sum_{<x,y>} \sigma(x)\sigma(y) - mH \sum_{x \in V} \sigma(x).$$

It is formal since we cannot operate with $H(\sigma)$.

Assume $H = 0$ doing this solely for the relative simplicity of the calculations made below. Let Λ be a finite subset of V . Assume $\Omega(\Lambda)$ is the set of all configuration $\sigma(\Lambda)$ on Λ, that is the functions $\{\sigma(x), x \in \Lambda\}$. Let $\overline{\sigma}(V \setminus \Lambda)$ be a fixed configuration on $V \setminus \Lambda$. The total energy or Hamiltonian of configuration $\sigma(\Lambda) \in \Omega(\Lambda)$ under condition $\overline{\sigma}(V \setminus \Lambda)$ is defined as

$$H(\sigma(\Lambda) \,|\, \overline{\sigma}(V \setminus \Lambda)) = -J \sum_{<x,y>, x,y \in \Lambda} \sigma(x)\sigma(y) - J \sum_{<x,y>, x \in \Lambda, y \notin \Lambda} \sigma(x)\sigma(y).$$

When all boundary points $\{\overline{\sigma}(y), y \in V \setminus \Lambda\}$ are fixed as $+1$, we have the positive boundary condition and when they are fixed as -1, we have negative boundary condition. The free boundary condition corresponds to the case the second sum in the above is absent, that is formally all boundary points are fixed as 0 .

We will consider a semi-infinite Cayley tree Γ_+^2 of order 2, i.e. an infinite graph without cycles with 3 edges issuing from each vertex except for x^0 which has only 2 edges.

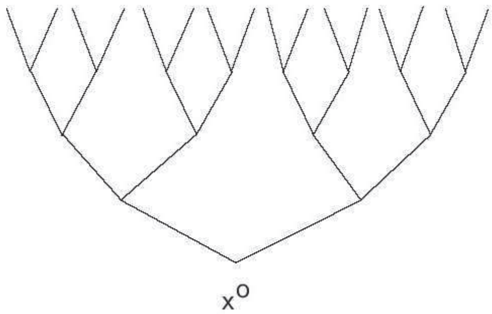

Figure 4: Semi-infinite Cayley tree Γ_+^2 of order 2.

Let Ω_n be a set of all configurations on V_n . Let us fix some n and let σ_n be a configuration on V_n . Now on infinite set $V_n^c = V \setminus V_n$ we will fix a configuration $\overline{\sigma}^n$ and call it a boundary condition. For any n with given boundary condition $\overline{\sigma}^n$ we have finitely many configurations on V . Now for given boundary condition $\overline{\sigma}^n$ energy of configuration σ_n is assigned by

$$H(\sigma_n \,|\, \overline{\sigma}^n) = -J \sum_{<x,y> \in V_n} \sigma(x)\sigma(y) - J \sum_{x \in V_n, y \in V_n^c} \sigma(x)\sigma(y). \tag{14}$$

Here the first and second sum are taken over all pairs $<x,y>$ of nearest neighbour. Now this energy is a finite number for for any σ_n. We define the *conditional Gibbs state* $\mu_n(\cdot|\overline{\sigma}^n$ on V_n with boundary condition $\overline{\sigma}^n$ and Hamiltonian (14) to be the state $\mu_n(\cdot|\sigma^n)$ given by

$$\mu_n(\sigma_n|\overline{\sigma}^n)=Z_n^{-1}(\overline{\sigma}^n)\exp(-\beta H(\sigma_n|\overline{\sigma}^n))$$

for any $\sigma_n\in\Omega_n$, where

$$Z_n(\overline{\sigma}^n)=\sum_{\sigma_n\in\Omega_n}\exp(-\beta H(\sigma_n|\overline{\sigma}^n))(16)$$

Thus we have a sequence of conditional Gibbs states $\{\mu_n(\cdot|\overline{\sigma}^n)\}$.

Below we prove that the positive and negative boundaries $\overline{\sigma}^n$ give different probabilities for event $\{\sigma\in\Omega:\sigma(x^0)=+1\}$ as n tends to infinity, i.e. we reach spontaneous magnetization on Cayley Tree.

Assume $\Omega_n^+=\{\sigma_n\in\Omega_n:\sigma_n(x^0)=+1\}$ and $\Omega_n^-=\{\sigma\in\Omega_n:\sigma(x^0)=-1\}$.

Then $\Omega_n=\Omega_n^+\cup\Omega_n^-$ and $\cup_n\Omega_n^+=\{\sigma\in\Omega:\sigma(x^0)=+1\}$,that is, Ω_n^+ tends to $\{\sigma\in\Omega:\sigma(x^0)=+1\}$ as n tends to infinity.

For fixed n we divide the partition function $Z_n(\overline{\sigma}^n)$ into two sums

$$Z_n(\overline{\sigma}^n)=Z_n^+(\overline{\sigma}^n)+Z_n^-(\overline{\sigma}^n)$$

where

$$Z_n^+(\overline{\sigma}^n)=\sum_{\sigma_n\in\Omega_n^+}\exp(-\beta H(\sigma_n|\overline{\sigma}^n))$$

and

$$Z_n^-(\overline{\sigma}^n)=\sum_{\sigma_n\in\Omega_n^-}\exp(-\beta H(\sigma_n|\overline{\sigma}^n))$$

Thus $Z_n^-(\overline{\sigma}^n)$ sums over all configurations in Ω_n^- which assign -1 to the root x^0, and $Z_n^+(\overline{\sigma}^n)$ sums over all configurations in Ω_n^+ which assign $+1$. We now compute the ratio of the probability of a -1 at the root to the probability of a $+1$ at the root,that is, $\frac{\mu_n(\Omega_n^-|\overline{\sigma}^n)}{\mu_n(\Omega_n^+|\overline{\sigma}^n)}$. This ratio we can rewrite as

$$\frac{\mu_n(\sigma_n(x^0)=-1|\overline{\sigma}^n)}{\mu_n(\sigma_n(x^0=1|\overline{\sigma}^n)}$$

This ratio, of course, determines the probability of a $+1$ at the root x^0 since the sum of the two probabilities is 1. We define

$$u_n=\frac{\mu_n(\sigma_n(x^0)=-1|\overline{\sigma}^n)}{\mu_n(\sigma_n(x^0)=1|\overline{\sigma}^n)}=\frac{Z_n^-}{Z_n^+}.$$

If we can find the limit of u_n as n tends to infinity, we will find the limiting ratio for the probability of a -1 to the probability of a $+1$ at the root, that is,

$$\frac{\mu(\sigma(x^0)=-1)}{\mu(\sigma(x^0)=1)}=\lim_{n\to\infty}\frac{\mu_n(\sigma_n(x^0)=-1|\overline{\sigma}^n)}{\mu_n(\sigma_n(x^0)=1|\overline{\sigma}^n)}.$$

We consider the possibilities for the first level W_1 of our tree with a $+1$ at the root x^0. There are now only three essentially different possibilities as shown in figure 5

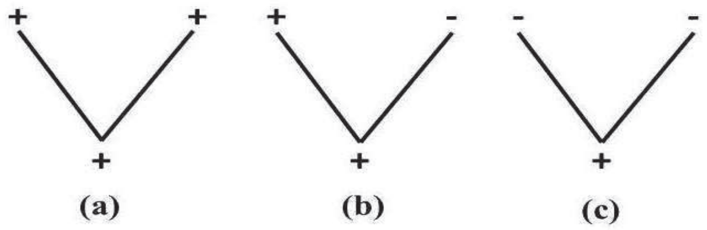

Figure 5: Possible configurations on W_1 with a $+1$ at the root x^0.

Then we have

$$u_n = \left[\frac{1+au_{n-1}}{a+u_{n-1}}\right]^2$$

The value u_1 may be obtained by considering a graph V_1. With positive boundary

$$u_1 = \frac{P\begin{pmatrix} ++ \\ - \end{pmatrix}}{P\begin{pmatrix} ++ \\ + \end{pmatrix}} = a^4$$

Similarly, with negative boundary

$$u_1 = \frac{P\begin{pmatrix} -- \\ - \end{pmatrix}}{P\begin{pmatrix} -- \\ + \end{pmatrix}} = a^{-4}$$

We can then compute u_n for $n > 1$ by using the fact that

$$u_n = f(u_{n-1})$$

where

$$f(x) = \left[\frac{1+ax}{a+x}\right]^2$$

We then ask if the sequence u_n a tends to limit u, and, if so, does this limit depend upon the value of u_1? It is not hard to show by simple calculus arguments that in the attractive case ($J > 0$),the sequence $\{u_n\}$ has a limit.We note that if there is a u such that $u_n \to u$, then we must have that $u = f(u)$,i.e., u is a fixed point of f. This follows from the fact that f is continuous and $u_n = f(u_{n-1})$. Let us describe fixed points of f, i.e., solve following equation

$$x = \left[\frac{1+ax}{a+x}\right]^2$$

or

$$(x-1)(x^2-(a^2-2a-1)x+1)=0$$

This equation has three positive roots $x^* < 1, x^{**} = 1, x^{***} > 1$, if $a > \sqrt{3}$, i.e., $T < T_c$, where $T_c = 4J/k\log 3$. It is easy to check that the smallest fixed point gives the limiting probability ratio for the positive boundary and the largest fixed point gives the limiting probability ratio for the negative random boundary. These measures correspond to the positive and negative boundaries respectively. Thus we have the effect of spontaneous magnetization.

Applications of Ising model

The Ising model continues to enjoy great success in a wide variety of applications. While Ising discussed only the magnetic interpretation, the same model has since been found applicable to a number of other physical and biological systems such as gases (we will consider later), binary alloys, and cell structures. A sociologically oriented application has been suggested by

Weidlich (1971). Here one considers a group of people, each of whom at a given moment is a "conservative" ("up") or a "liberal" ("down"). The energy (1) might better be called "tension". The first term in (1) is the tension caused by people interacting. The external field represents, for example, the current state of the government, liberal or conservative. Minimum tension (maximum boredom) occurs if all people agree and agree with government.

Current State of Ising model

The axial next-nearest-neighbour Ising (ANNNI) model defined on regular lattices Z^d originally introduced by Elliot (1961) to describe the sinusoidal magnetic structure of Erbium, has been studied extensively by a variety of techniques. A particularly interesting and powerful method is the study of modulated phases through the measure-preserving map generated by the mean-field equations, as applied by Bak (1981 and 1982) and Jensen and Bak (1983) to the ANNNI model. The main drawback of the method lies in the fact that thermodynamic solutions correspond to stationary but unstable orbits. However, when these models are defined on Cayley tree of order k , i.e., a graph without cycles with exactly $k+1$ edges issuing from each vertex, as in the case of the Ising model with competing interactions examined by Vannimenus (1981), it turns out that physically interesting solutions correspond to the attractors of the mapping. This simplifies the numerical work considerably, and detailed study of the whole phase diagram becomes feasible. Apart from the intrinsic interest attached to the study of models on trees, it is possible to argue that the results obtained on trees provide a useful guide to the more involved study of their counterparts on crystal lattices. The existence of competing interactions lies at the heart of a variety of original phenomena in magnetic systems, ranging from the spin-glass transitions found in many disordered materials to the modulated phases with an infinite number of commensurate regions, that are observed in certain models with periodic interactions (Bak and Von Boehm 1980), (Fisher and Selke 1980). A strong motivation to study competition effects on Cayley trees one can find in Vannimenus paper (Vannimenus 1981). Indeed, for many problems the solution on a tree is much simpler than on a regular lattice and is equivalent to the standard Bethe-Peierls theory (Katsura and Takizawa 1974). Vannimenus proved numerically that the phase diagram of the model (Vannimenus 1981) contains a modulated phase, as found for similar models on periodic lattices.

The Ising model on a Cayley tree of order k with competing interactions has recently been studied extensively because of the appearance of nontrivial magnetic orderings (Vannimenus 1981) (see also references in (Vannimenus 1981)). Generalizations of the Vannimenus model that include links connecting couplings at distance 2 (without restriction) as well as the presence of an external magnetic field, have been studied in detail in Mariz, Sallis, and Albuquerque (1985), the model that includes also links connecting couplings at distance 3 have been studied in Silva and Coutinho 1986), the Vannimenus model on a Cayley tree of arbitrary order have been studied in Ganikhodjaev and Uguz (2011).

Conclusion

All that we have seen so far shows us one clear fact: the Qur'an is such a book that all the news related in it has proved to be true. Facts about scientific subjects and the news given about the future, facts that no one could have known at the time, were announced in its verses. It is impossible for this information to have been known with the level of knowledge and technology of the day. It is clear that this provides clear evidence that the Qur'an is not the word of man. The Qur'an is the word of the Almighty Allāh, the Originator of everything and the One Who encompasses everything with His knowledge. In one verse, Allāh says in the Qur'an

> *"If it had been from other than Allāh, they would have found many inconsistencies in it."*
>
> *(The Qur'an 4:82)*

Not only are there no inconsistencies in the Qur'an, but every piece of information it contains reveals the miracle of this divine book more and more each day. What falls to man is to hold fast to this divine book revealed by Allāh, and receive it as his one and only guide. In one of the verses, Allāh calls out to us:

"And this is a Book which We have revealed as a blessing: so follow it and be righteous, that ye may receive mercy"

(The Qur'an 6:155)

In another verses, Allāh remarks:

> *"Say, The truth is from your Lord": Let him who will believe, and let him who will, reject (it): for the wrong-doers We have prepared a Fire whose (smoke and flames), like the walls and roof of a tent, will hem them in: if they implore relief they will be granted water like melted brass, that will scald their faces, how dreadful the drink! How uncomfortable a couch to recline on."*

(The Qur'an 18:29)

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