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Comparison of homotopy analysis method and homotopy-perturbation method for purely nonlinear fin-type problems

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Abstract

In this paper, the homotopy analysis method (HAM) is compared with the homotopy-perturbation method (HPM) and the Adomian decomposition method (ADM) to determine the temperature distribution of a straight rectangular fin with power-law temperature dependent surface heat flux. Comparisons of the results obtained by the HAM with that obtained by the ADM and HPM suggest that both the HPM and ADM are special case of the HAM.

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Keywords: Homotopy analysis method; Homotopy-perturbation method; Nonlinear DEs; Fins

1. Introduction

In this paper, we consider the temperature distribution along a fin of constant cross-sectional area and thermal conductance at the fin base. In dimensionless form, the problem reduces to the nonlinear boundary-value problem of a one-dimensional steady-state heat conduction equation for the temperature distribution along a straight fin, cf. [1,2],

$$y'' - My^m = 0, \tag{1}$$

where the prime denotes differentiation w.r.t. x and the dimensionless variable x is measured from the fin tip, y is the temperature, M is the convective–conductive parameter of the fin and the exponent m depends on the heat transfer mode (cf. [2,3]). For practical interest the physical values of m are 5/4 and 4/3 for free convection,

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three for nucleate boiling and four for radiation [1]. For simplicity, we consider the case $M = 1$, the fin tip is isolated and therefore the boundary conditions to Eq. (1) can be written as

$$y'(0) = 0, \quad y(1) = 1. \tag{2}$$

Fins are extensively employed to enhance the heat transfer between a solid surface and its convective, radiative, or convective radiative surface [4]. In many applications, various heat transfer modes, such as convection, nucleate boiling, transition boiling, and film boiling, the heat transfer coefficient are no longer uniform. A fin with an insulated end has been studied by many investigators [5–9]. Most of them are immersed in the investigation of single boiling mode on an extended surface. Recently, Lesnic [1] applied the standard Adomian decomposition method to determine analytically the temperature distribution within a single fin with a temperature dependent heat transfer coefficient. Liu [10] found that the Adomian method could not always satisfy all boundary conditions leading to an error at boundaries. The ADM also suffers the drawback of the need to calculate the Adomian polynomials which can be difficult. Another analytical method called homotopy analysis method (HAM) first proposed by Liao in his Ph.D. dissertation in 1992 [11] improved by Liao [12–16] can be a simple alternative. The application of HAM in nonlinear problems has been presented by many researchers, cf. [17–21]. In particular, the HAM was employed for solving the Blasius viscous flow problems [22], nonlinear problems [23], generalized Hirota–Satsuma coupled KdV equation [24], non-homogeneous Blasius problem [25], unsteady boundary-layer flows over a stretching flat plate [26], decaying boundary layers as limiting cases of families of algebraically decaying ones [27], thin flim flow of non-Newtonian fluids on a moving belt [28] and derivation of the Adomian decomposition method [29].

In this paper, we apply the homotopy analysis method (HAM) to obtain more accurate temperature distribution within a single fin with a temperature dependent heat transfer coefficient. For comparison purpose, we also obtain homotopy-perturbation method (HPM) solutions. The results are compared with the available exact and the Adomian decomposition solutions of [1].

2. Solution procedure

2.1. HAM solutions

The basic ideas of HAM described in [12–16]. From Eq. (1) and choosing $y(0) = C$, the solution of Eq. (1) can be expressed by a set of base functions

$$\{x^n | n = 1, 2, \ldots\},$$

in the form

$$y(x) = \sum_{n=0}^{\infty} d_n x^{2n}, \tag{3}$$

where d_n are to be determined and $C \in (0, 1)$ an unknown arbitrary constant representing the temperature at the fin tip is to be determined by imposing the second boundary condition given by Eq. (2).

In HAM, we have the so-called *rule of solution expression*, i.e. the solution of (1) must be expressed in the same form as (3). Under the first *rule of solution expression* and according to the condition $y(0) = C$, it is straightforward to choose the initial approximation of $y(x)$ as

$$y_0(x) = C,$$

and an auxiliary linear operator as

$$L[\phi(x; q)] = \frac{\partial^2 \phi(x; q)}{\partial x^2},$$

with the property

$$L[C_1 + C_2 x] = 0,$$

where C_i $(i = 1, 2)$ are constants. From (1), we define a nonlinear operator as

$$N[\phi(x;q)] = \frac{\partial^2 \phi(x;q)}{\partial x^2} - M\phi(x;q)^m.$$

Using the above definition, we construct the zeroth-order deformation equations

$$(1-q)L[\phi(x;q) - y_0(x)] = q\hbar H(x)N[\phi(x;q)] \tag{4}$$

with the boundary conditions

$$\phi(0;q) = C, \quad \left.\frac{\partial \phi(x,q)}{\partial x}\right|_{x=0} = 0,$$

where $q \in [0,1]$ is an embedding parameter, $\hbar$ is a non-zero auxiliary function, L is an auxiliary linear operator, $y_0(x)$ is an initial guess of $y(x)$ and $\phi(x;q)$ is an unknown function. Obviously, when $q = 0$ and $q = 1$,

$$\phi(x;0) = y_0(x), \quad \phi(x;1) = y(x).$$

Therefore, the embedding parameter q increases from 0 to 1, $\phi(x;q)$ varies from the initial guess $y_0(x)$ to the solution $y(x)$. Expanding $\phi(x;q)$ in Taylor series with respect to q one has

$$\phi(x;q) = y_0(x) + \sum_{n=1}^{+\infty} y_n(x)q^n,$$

where

$$y_0(x) = \frac{1}{n!}\left.\frac{\partial^n \phi(x,q)}{\partial q^n}\right|_{q=0}.$$

If the auxiliary linear operator, the initial guess and the auxiliary parameters $\hbar$ are so properly chosen, the above series is convergent at $q = 1$, then

$$y(x) = y_0(x) + \sum_{n=1}^{+\infty} y_n(x),$$

must be one of the solutions of the original nonlinear equation (1), as proved by Liao [12]. Now, we define the vector

$$\vec{y}_{m-1} = \{y_0(x), y_1(x), \ldots, y_{m-1}(x)\}.$$

The nth-order deformation equation is

$$L[y_n(x) - \chi_n y_{n-1}(x)] = \hbar H(x)R_n[\vec{y}_{n-1}(x)], \tag{5}$$

with the boundary conditions

$$y_n(0) = y_n'(0) = 0, \tag{6}$$

where

$$R_1[\vec{y}_0(x)] = y_0'' - My_0^m,$$
$$R_2[\vec{y}_1(x)] = y_1'' - Mmy_0^{m-1}y_1,$$
$$R_3[\vec{y}_2(x)] = y_2'' - M\left[my_0^{m-1}y_2 + \frac{m(m-1)}{2}y_0^{m-2}y_1^2\right],$$
$$R_4[\vec{y}_3(x)] = y_3'' - M\left[my_0^{m-1}y_3 + m(m-1)y_0^{m-2}y_1y_2 + \frac{m(m-1)(m-2)}{6}y_0^{m-3}y_1^3\right],$$

etc., where the primes denote differentiations with respect to x and

$$\chi_n = \begin{cases} 0, & j \leqslant 1, \\ 1, & j > 1. \end{cases}$$

Now, the solution of the nth-order deformation Eq. (5) for $n \geqslant 1$ becomes

$$y_n(x) = \chi_n y_{n-1}(x) + \hbar \int_0^x \int_0^x R_n(\vec{y}_{n-1})\, \mathrm{d}\tau\, \mathrm{d}\tau + c_1 + x c_2,$$

where the integration constants $c_i\,(i = 1, 2)$ are determined by the boundary conditions (6). By the rule of coefficient ergodicity, we can obtain, uniquely, the corresponding auxilary function $H(x) = 1$. Therefore, we have

$$\begin{aligned}
y_0(x) &= C, \\
y_1(x) &= -\frac{1}{2} C^m \hbar M x^2, \\
y_2(x) &= \frac{1}{6} C^m \hbar M \left[-3(1+\hbar)x^2 + \frac{1}{4} C^{m-1} \hbar M m x^4 \right], \\
y_3(x) &= -\frac{1}{120} C^m \hbar M \left[60(1+\hbar)^2 x^2 - 10 C^{m-1} \hbar M (1+\hbar) m x^4 + \frac{1}{6} C^{2m-2} \hbar^2 M^2 (4m^2 - 3m) x^6 \right],
\end{aligned}$$

etc.

It is important to note that, in HAM has great freedom to choose the auxiliary linear operator, the initial guess and the auxiliary parameters $\hbar$. When we choose $\hbar = -1$ and successively obtain

$$\begin{aligned}
y_0(x) &= C, \\
y_1(x) &= \frac{1}{2} C^m M x^2, \\
y_2(x) &= \frac{1}{24} M^2 C^{2m-1} m x^4, \\
y_3(x) &= \frac{1}{720} M^3 C^{3m-2} (4m^2 - 3m) x^6,
\end{aligned} \tag{7}$$

etc., which are exactly same as the ADM solution [1]. Hence the nth-order approximation can be expressed by

$$y(x) \approx \sum_{k=0}^{n} y_k(x) = \sum_{i=0}^{n} d_i(h) x^i, \tag{8}$$

is a family of solution expression, in the auxiliary parameter $\hbar$, which recover the ADM solution [1] as a special case $\hbar = -1$.

The complete solution (1) as given in (8) is obtained once the constant C is determined by imposing the second boundary condition given by Eq. (2). Note that the value of C must lie in the interval $(0, 1)$ to represent

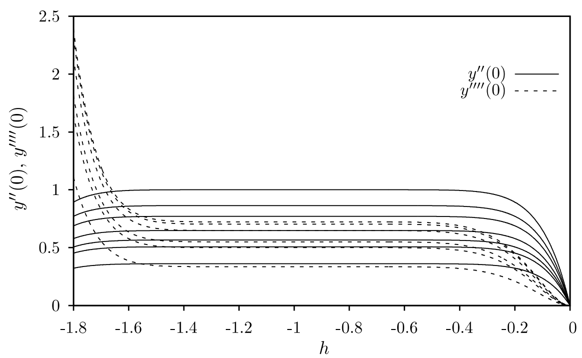

Fig. 1. The $\hbar$-curves of the 10th-order HAM approximations $y''(0)$ and $y''''(0)$ for several values of m (8) when $H(x) = 1$.

the temperature at the fin tip. But, first we need to determine the range of values of $\hbar$ for which the series solution (8) is convergent. To do so, we plot the so-called $\hbar$-curves of the 10th-order HAM approximation for $y''(0)$ and $y''''(0)$ for several values of m and $H(x) = 1$ as shown in Fig. 1. We can see that roughly $-1.5 < \hbar < -0.3$. So, now having determined the range of $\hbar$, we can proceed to find C.

2.2. HPM solutions

According to the so-called homotopy-perturbation method (HPM) [30], we construct a homotopy $v(r,p) : \Omega \times [0,1] \to \Re$ in Eq. (1) which satisfies

$$y'' - y_0'' + p(y_0'' - My^m) = 0, \tag{9}$$

where $y_0(x)$ is the initial guess approximation.

Suppose the solution of Eq. (1) has the form:

$$y(x) = u_0(x) + pu_1(x) + p^2u_2(x) + \cdots, \tag{10}$$

and choose the initial approximation as

$$u_0(x) = y_0(x) = y(0) = C, \tag{11}$$

where C is to be determined imposing the second boundary condition given by Eq. (2). Substituting (10) into (9) and equating the terms with identical powers of p, we get

$$\frac{d^2u_1}{dx^2} + \frac{d^2y_0}{dx^2} - Mu_0^m = 0, \quad u_1(0) = 0, \quad \frac{du_1}{dx}(0) = 0, \tag{12}$$

$$\frac{d^2u_2}{dx^2} - Mmu_1u_0^{m-1} = 0, \quad u_2(0) = 0, \quad \frac{du_2}{dx}(0) = 0, \tag{13}$$

$$\frac{d^2u_3}{dx^2} - Mmu_2u_0^{m-1} - \frac{1}{2}Mm(m-1)u_1^2u_0^{m-2} = 0, \quad u_3(0) = 0, \quad \frac{du_3}{dx}(0) = 0, \tag{14}$$

etc.

Solving (12)–(14), we have

$$u_0(x) = C, \tag{15}$$

$$u_1(x) = \frac{1}{2}MC^mx^2, \tag{16}$$

$$u_2(x) = \frac{1}{24}M^2mC^{2m-1}x^4, \tag{17}$$

$$u_3(x) = \frac{1}{720}M^3m(4m-3)C^{3m-2}x^6, \tag{18}$$

etc., which are also same as the HAM solutions for special case $\hbar = -1$, $H(x) = 1$.

Table 1
Comparison between the 5-term HAM and the exact solutions for $m = 1$ and $M = 1$

x	Exact	5-term HAM
0.0	0.648054274	0.648054390
0.1	0.651297246	0.6512973633
0.2	0.661058620	0.661058739
0.3	0.677436092	0.677436213
0.4	0.700593571	0.700593697
0.5	0.730762826	0.730762957
0.6	0.768245800	0.768245938
0.7	0.813417638	0.813417780
0.8	0.866730432	0.866730569
0.9	0.928717757	0.928717861
1.0	1.000000000	1.000000000

3. Discussion

In Table 1, we show the comparisons between the 5-term HAM solutions and the exact solutions for the case $m = 1$ and $M = 1$. We observe that the results of the 5-term HAM is very close to the exact solutions which confirm the validity of the HAM. Fig. 2 shows the temperature profiles obtained by HAM for several assigned values of m for the case $M = 1$. All the numerical results obtained by the 5-term HAM are exactly same as the ADM [1] solutions and HPM solutions for special case $\hbar = -1$, $H(x) = 1$. So its means that the HPM and the ADM is a special case of HAM. But HAM is more general and contains the auxiliary parameter $\hbar$, which provides us with a simple way to adjust and control the convergence region of solution series. As pointed out by Abbasbandy in [18] one had to choose a proper value of $\hbar$ to ensure the convergence

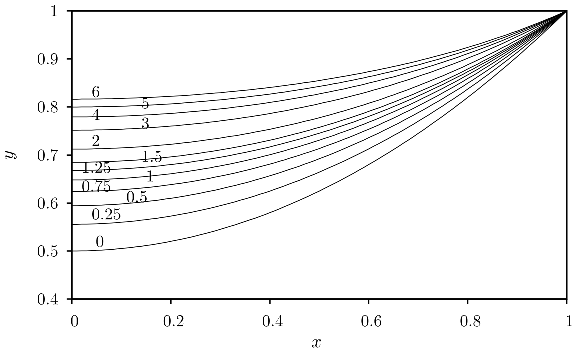

Fig. 2. 5-term HAM solutions for different values of m and $M = 1$, $\hbar = -1$, $H(x) = 1$.

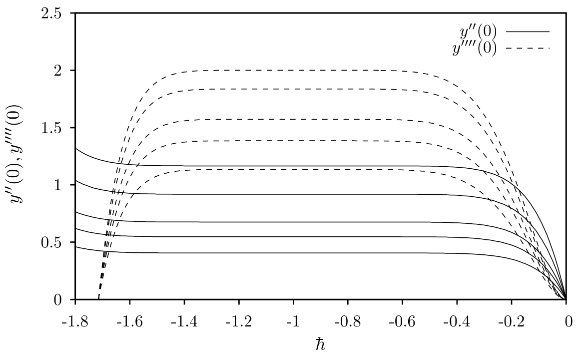

Fig. 3. The $\hbar$-curves of the 10th-order HAM approximations $y''(0)$ and $y''''(0)$ for several values of m (8) when $H(x) = 1$ and $M = 2$.

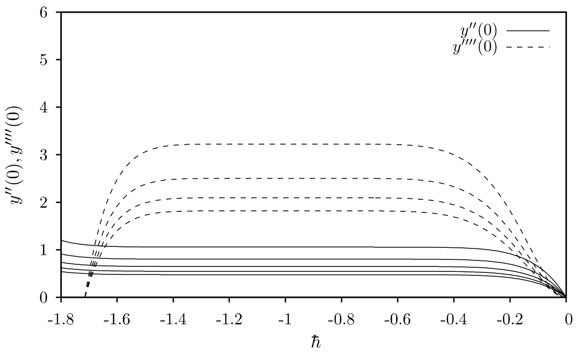

Fig. 4. The $\hbar$-curves of the 10th-order HAM approximations $y''(0)$ and $y''''(0)$ for several values of m (8) when $H(x) = 1$ and $M = 5$.

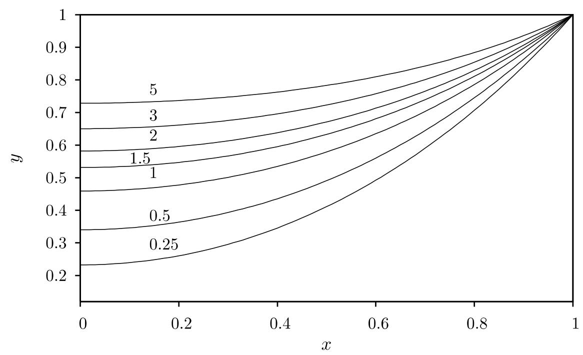

Fig. 5. 5-term HAM solutions for different values of m and $M = 2$, $\hbar = -0.9$, $H(x) = 1$.

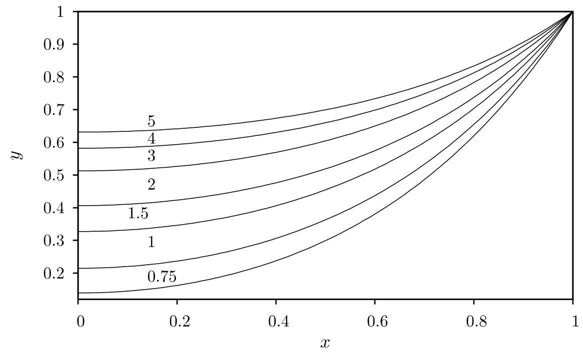

Fig. 6. 5-term HAM solutions for different values of m and $M = 5$, $\hbar = -0.8$, $H(x) = 1$.

of series solution for strongly nonlinear problems and the pertubation method and HPM are not valid for larger value of M. For the larger value of M Eq. (1) corresponds to strong nonlinearity. So it is more important to show that the HAM gives convergent series solution for any larger values of M by choosing proper values of $\hbar$. To do so, we consider $M = 2$ and $M = 5$ which were not considered in [1]. To choose the proper value of $\hbar$, we plot the $\hbar$-curves of the 10th-order HAM approximation for $y''(0)$ and $y''''(0)$ for several values of m and $H(x) = 1$ in Figs. 3 and 4, respectively for $M = 2$ and $M = 5$. Figs. 5 and 6 represent the 5-term HAM solutions for different values of m when $\hbar = -0.9$, $H(x) = 1$, $M = 2$ and $\hbar = -0.8$, $H(x) = 1$, $M = 5$. Similarly, we can get convergent HAM series solution by choosing the proper values of $\hbar$ for any larger values of M.

4. Conclusions

In this paper, the power-law fin-type problem was solved via the homotopy analysis method (HAM). The obtained solutions are more accurate with easily computable terms. Also in HAM we can choose $\hbar$ in appropriate way to ensure the convergence of series solution for strongly nonlinear problems. Comparison with the exact solution and decomposition method shows that the homotopy analysis method is a promising tool for finding approximate analytical solutions to strongly nonlinear problems.

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