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Numerical Method for Inverse Laplace Transform with Haar Wavelet Operational Matrix

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ABSTRACT

Wavelets have been applied successfully in signal and image processing. Many attempts have been made in mathematics to use orthogonal wavelet function as numerical computational tool. In this work, an orthogonal wavelet function namely Haar wavelet function is considered. We present a numerical method for inversion of Laplace transform using the method of Haar wavelet operational matrix for integration. We proved the method for the cases of the irrational transfer function using the extension of Riemenn-Liouville fractional integral. The proposed method extends the work of J.L.Wu et al. (2001) to cover the whole of time domain. Moreover, this work gives an alternative way to find the solution for inversion of Laplace transform in a faster way. The use of numerical Haar operational matrix method is much simpler than the conventional contour integration method and it can be easily coded. Additionally, few benefits come from its great features such as faster computation and attractiveness. Numerical results demonstrate good performance of the method in term of accuracy and competitiveness compare to analytical solution. Examples on solving differential equation by Laplace transform method are also given.

| operational matrix | numerical inversion | Inversion of Laplace transform | Haar wavelet |

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1. INTRODUCTION

Laplace transforms is known to be an important tool in solving mathematical equations that arise in engineering problem. Since its discovery by a French mathematician, it has been widely applied and continuously researched by scholars from various fields. Those scholars had put through enormous amount of efforts in finding its inverse function numerically and analytically. This is because finding the inverse of Laplace transform is considered to be a difficult task due to its limitation in the inversion table of inverse Laplace transform,in the sense that it couldn'tcater most of the engineering problems which always associated with complexity of mathematical equation.

The objective of this paper is to propose a numerical inversion of Laplace transform using Haar operation matrix. The proposed method in this paper is an extension work of J.L. Wu et al (2001) that covers the whole time domain in finding inversion Laplace transform numerically using Haar wavelet operational matrix for integration. J.L. Wu et al. has proposed a new unified method to derive the operational matrix of any orthogonal functions for integration within the interval of $0 \le t < 1$. We derive the Haar operationalmatrix based on Wu et al. works but extending it using generalised block pulse function operational matrix for integration [2,7] which is convenient as it will fit the expansion of Haar series for $0 \le t < \tau$.

Before Haar wavelet operational matrices were used to find inversion of Laplace transform, there are other literatures that used other orthogonal functions as well. In 1977, C. F. Chen et al have been using Walsh operational matrices for solving various distributed-parameters systems such as heat conduction and percolation problem [8]. Later, a more rigorous approach has been taken by Wang Chi-Hsu to derive the generalised block pulse operational matrices [7].According to Wang, inversions of Laplace transform for rational and irrational transfer function illustrated by using generalized block pulse operational matrices is proven to be more accurate compare to previous work by Chen[8].

2. MATHEMATICAL REVIEW

2.1 Haar Wavelet Function

An analytic function $f(t)$ can be expandedin a series

$$f(t) = \sum_{n=0}^{\infty} a_n \psi_n(t) \tag{2.1}$$

where $\varphi_n(t)$ is the basis in the Hilbert space $L^2(R)$ and a_n is coefficient of the series. The coefficients can be obtained as follows,

$$a_n = \int_{-\infty}^{\infty} f(t)\psi_n(t)dt \tag{2.2}$$

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For example, if we have a function $\psi_n(t) = t^n$, we could expand the function using power series expansion such as Taylor series expansion. Same goes to a function with sinusoidal basis, we could use Fourier series expansion. In this work an orthogonal function namely Haar wavelet function is considered. The set of this function is a group of square waves in intervals of $[0, \tau)$ and defined as below

$$h_0(t) = \frac{1}{m^{1/2}} \qquad (0 \le t < \tau) \tag{2.3}$$

$$h_1(t) = \frac{1}{m^{1/2}} \begin{cases} 1 & \xi_1 \le t < \xi_2 \\ -1 & \xi_2 \le t < \xi_3 \\ 0 & \text{elsewhere} \end{cases} \tag{2.4}$$

$$h_i(t) = \frac{1}{m^{1/2}} \begin{cases} 2^{j/2} & \xi_1 \le t < \xi_2 \\ -2^{j/2} & \xi_2 \le t < \xi_3 \\ 0 & \text{elsewhere} \end{cases} \tag{2.5}$$

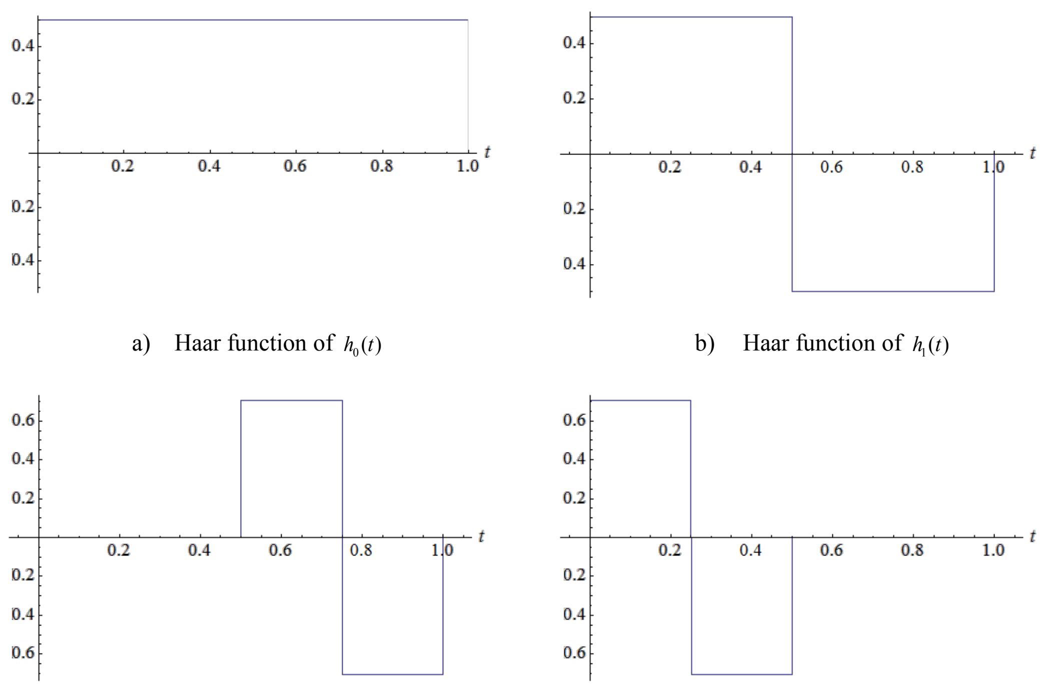
where $\xi_1 = (k - 1/2^j)\tau$, $\xi_2 = ((k-1/2)/2^j)\tau$, $\xi_3 = (k/2^j)\tau$ $m = 2^J$, $i = 0, 1, 2, \cdots, m-1$ and the resolution J is a positive integer. While j and k denoted the integer decomposition of the index i, for example $i = 2^j + k - 1$ in which $k = 1, 2, 3, \cdots, 2^j$. $h_0(x)$ is defined as a constant and called scaling function, while $h_1(x)$ is called mother wavelet function or fundamental square wave. All the others following Haar wavelet functions are generated from mother wavelet function, $h_1(t)$ with translation and dilation process.

$$h_i(t) = 2^{j/2} h_1(2^j t - k) \tag{2.6}$$

where $i = 2^j + k - 1$, $j \ge 0, 0 \le k < 2^j$.Haar wavelet function also is an orthogonal function, so that it holds the property as below

$$(h_p(t), h_n(t)) = \int_0^t h_p(t) h_n(t)\, dt = \begin{cases} \tau/m & \text{if} \quad p = n \\ 0 & \text{if} \quad p \ne n \end{cases} \tag{2.7}$$

The orthogonal set of the first four Haar function $(m = 4)$ in the interval of $(0 \le t < 1)$ can be shown in Figure 1 below.



a) Haar function of $h_0(t)$

b) Haar function of $h_1(t)$

c) Haar function of $h_2(t)$

d) Haar function of $h_3(t)$

Fig. 1 First four Haar function

2.2 Haar Series Expansion

Haar wavelet function is not continuous. As for Haar series expansion, any function $x(t)$ can be decomposed into Haar series and can be written as

$$x(t) = \sum_{i=0}^{\infty} c_i h_i(t) \tag{2.8}$$

If the function $x(t)$ may be approximated as a piecewise constant then the sum in equation (2.8) may be truncated after m terms and defined within interval $0 \le t < \tau$, then it becomes,

$$x(t) \approx \sum_{i=0}^{m-1} c_i h_i(t) \tag{2.9}$$

where the Haar coefficient c_i are determined by

$$c_i = \frac{m}{\tau} \int_0^t x(t) h_i(t) dt \tag{2.10}$$

2.3 Haar Wavelet Matrix

Equation (2.8) can be expressed in matrix form as

$$\mathbf{x}_m(t) = \mathbf{c}_m^T \mathbf{H}_m(t) \tag{2.11}$$

$\mathbf{x}_m(t)$ denotes the truncated sum which is expansion of $0 \le t < \tau$ and m is the size of the $m \times m$ matrix. Haar coefficient vector $\mathbf{c}_m^T$ and Haar function vector, $\mathbf{H}_m(t)$ are defined as

$$\begin{aligned} \mathbf{c}_m^T &= \begin{bmatrix} c_0 & c_1 & \cdots & c_{m-1} \end{bmatrix} \\ \mathbf{H}_m(t) &= \begin{bmatrix} h_0(t) & h_1(t) & \cdots & h_{m-1}(t) \end{bmatrix}^T \end{aligned} \tag{2.12}$$

Taking the collocation points as following

$$t_i = \frac{2i-1}{2m}\tau, \quad i = 1, 2, \cdots, m \tag{2.13}$$

It is defined that the m square Haar wavelet matrix, $\mathbf{H}_m$ as

$$\mathbf{H}_m = \left[\mathbf{H}_m\left(\frac{\tau}{2m}\right) \quad \mathbf{H}_m\left(\frac{3\tau}{2m}\right) \quad \cdots \quad \mathbf{H}_m\left(\frac{(2m-1)\tau}{2m}\right) \right] \tag{2.14}$$

For instance, the fourth Haar wavelet matrix $(m=4)$, $\mathbf{H_4}$ in the interval of $0 \le t < 1$ can be represented in matrix form as below.

$$\begin{aligned} \mathbf{H}_4 &= \begin{bmatrix} h_0(1/8) & h_0(3/8) & h_0(3/8) & h_0(7/8) \\ h_1(1/8) & h_1(3/8) & h_1(3/8) & h_1(7/8) \\ h_2(1/8) & h_2(3/8) & h_2(3/8) & h_2(7/8) \\ h_3(1/8) & h_3(3/8) & h_3(3/8) & h_3(7/8) \end{bmatrix} \\ &= \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \end{aligned} \tag{2.15}$$

Haar wavelet is an orthogonal functions and it can be shown that

$$\mathbf{H}_m^{-1} = \mathbf{H}_m^T \tag{2.16}$$

by this method it is convenient to find the coefficient without performing the integration as equation (2.10)

$$\mathbf{c} = \mathbf{H}\mathbf{x}^T \tag{2.17}$$

Where $\mathbf{x}$ is a vector of a function $x(t)$ at the collocation point as equation (2.13).

2.4 Integration of Haar Wavelet Function and its Operational Matrix

Let consider the integration of a Haar wavelet function $H_m(t)$ given by

$$\int_0^t H_m(t_1) dt_1 = \mathbf{Q}_m H_m(t) \qquad 0 \le t < \tau \tag{2.18}$$

where $\mathbf{Q}_m$ is the generalised Haar operational matrix for integration of Haar wavelet function, $H_m(t)$ .We can write $H_m(t)$ in this form

$$H_m(t) = \mathbf{H}_m B_m(t) \tag{2.19}$$

where $B_m(t)$ is the block pulse function [2]

$$B_m(t) = \begin{cases} 1 & \psi_1 \le t < \psi_2 \\ 0 & \text{elsewhere} \end{cases} \tag{2.20}$$

where $\psi_1 = [(i-1)/m]\tau$ and $\psi_2 = (i/m)\tau$, for $i = 1, 2, \ldots, m$ which defined on the interval $(0, \tau]$ thus equation (2.18) can be written as

$$\int_0^t H_m(\tau) d\tau = \int_0^t \mathbf{H}_m B(\tau) d\tau = \mathbf{H}_m \int_0^t B_m(\tau) d\tau \tag{2.21}$$

It is known that the integration of block pulse function can be calculated as below

$$\int_0^t B_m(\tau) d\tau \cong \mathbf{F}_{\alpha m} B_m(t) \tag{2.22}$$

where $\mathbf{F}_{\alpha m}$ is taken from generalize blockpulse operational matrix for integrationwith $\alpha = 1$ and $b = \tau$ $(0 \le t < \tau)$ [7].

$$\mathbf{F}_{1m} = \frac{\tau}{2m} \begin{bmatrix} 1 & 2 & \cdots & 2 \\ 0 & 1 & \cdots & \vdots \\ \vdots & 0 & \ddots & 2 \\ 0 & \cdots & 0 & 1 \end{bmatrix}_{m \times m} \tag{2.23}$$

From equation (2.21)and (2.22)we obtain

$$\int_0^t H_m(\tau) d\tau = \mathbf{H}_m \mathbf{F}_{1m} B_m(t) \tag{2.24}$$

$B_m(t)$ in equation (2.24) is an identity matrix and can be neglected, then we have

$$\int_0^t H_m(\tau) d\tau = \mathbf{H}_m \mathbf{F}_{1_m} \tag{2.25}$$

The right hand side of equation (2.18) and (2.24) are identical, so that we obtain

$$\mathbf{Q}_m H_m(t) = \mathbf{H}_m \mathbf{F}_{1m} B_m(t) \tag{2.26}$$

Taking the collocation points as equation (2.13), we can write equation (2.26)as

$$\begin{aligned} \mathbf{Q}_m \mathbf{H}_m &= \mathbf{H}_m \mathbf{F}_{1m} \mathbf{I}_m \\ \mathbf{Q}_m &= \mathbf{H}_m \mathbf{F}_{1m} \mathbf{H}_m^T \end{aligned} \tag{2.27}$$

Thus we have generalised Haar operational matrix. For examplegeneralised Haar operational matrix when $m = 4$ and $\tau = 1$, from equation (2.27)we will have the matrix as below

$$\begin{aligned} \mathbf{Q}_4 &= \mathbf{H}_4 \mathbf{F}_{14} \mathbf{H}_4^T \\ &= \begin{bmatrix} 1/2 & -1/4 & -1/8\sqrt{2} & -1/8\sqrt{2} \\ 1/4 & 0 & -1/8\sqrt{2} & 1/8\sqrt{2} \\ 1/8\sqrt{2} & 1/8\sqrt{2} & 0 & 0 \\ 1/8\sqrt{2} & -1/8\sqrt{2} & 0 & 0 \end{bmatrix} \end{aligned} \tag{2.28}$$

Besides that the generalised Haar operational matrix for integration, $\mathbf{Q}_m$ also can be obtained from recursive formula by Chen Hsiao et. al [5] aftersome modifications were made to cover the interval of $[0,\tau)$. The generalised Haar operational matrix from recursive formula can be calculated by equation as below.

$$\mathbf{Q}_m = \frac{1}{2m}\begin{bmatrix} 2m\mathbf{Q}_{m/2} & -\tau\mathbf{H}^T_{m/2} \\ \tau\mathbf{H}^T_{m/2} & 0_{m/2} \end{bmatrix} \tag{2.29}$$

2.5 Riemann-Liouville Fractional Integral and Haar Wavelet Function

It is known that for integer n, the iterated integration with $(n-1)$ fold can be written asa single integral. It is a generalization form for natural order integrationand expressed as below [6],

$$(I^n f)(t) = \int_0^t \cdots \int_0^{t_3}\int_0^{t_2} f(t_1)\,dt_1\,dt_2 \ldots dt_n = \int_0^t \frac{(t-t_n)^{n-1}}{(n-1)!} f(t_n)\,dt_n \tag{2.30}$$

From the definition of integration of Haar wavelet function $H_m(t)$ in equation (2.18), and using the definition of equation (2.30), yields

$$(I^n H_m)(t) = \frac{1}{(n-1)!}\int_0^t (t-t_1)^{n-1} H_m(t_1)dt_1 \approx \mathbf{Q}^n_m H_m(t) \tag{2.31}$$

Generalization can be made to deal with fractional integral by substituting $(n-1)!$ with Gamma function $\Gamma(\alpha)$ [6], thus equation (2.31) become

$$(I^\alpha H_m)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-t_1)^{\alpha-1} H_m(t_1)dt_1 \approx \mathbf{Q}^\alpha_m H_m(t) \tag{2.32}$$

This is Riemann-Liouville fractional integral of a Haar wavelet function $H_m(t)$ with integral of order $\alpha > 0$ .Some modification is necessary to accommodate with expression in finding inversion of Laplace transform later. Firstly, we consider the fractional integral of Haar wavelet scaling function, $h_0(t)$ of order $\alpha > 0$ and equation (2.32) is then become,

$$\begin{aligned}(I^\alpha h_0)(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-t_1)^{\alpha-1} h_0(t_1)dt_1 \\ &= \frac{1}{\Gamma(\alpha)}\frac{1}{\sqrt{m}}\int_0^t (t-t_1)^{\alpha-1} dt_1 \\ &= \frac{1}{\sqrt{m}}\frac{t^\alpha}{\Gamma(\alpha+1)}\end{aligned} \tag{2.33}$$

by cross multiplying the above equation, yields

$$\begin{aligned}\frac{t^\alpha}{\Gamma(\alpha+1)} &= \frac{\sqrt{m}}{\Gamma(\alpha)}\int_0^t (t-t_1)^{\alpha-1} h_0(t_1)dt_1 \\ &= \frac{\sqrt{m}}{\Gamma(\alpha)}\int_0^t (t-t_1)^{\alpha-1}\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix} H_m(t_1)dt_1 \\ &= \mathbf{e}\frac{1}{\Gamma(\alpha)}\int_0^t (t-t_1)^{\alpha-1} H_m(t_1)dt_1\end{aligned} \tag{2.34}$$

where $\mathbf{e} = \begin{bmatrix}\sqrt{m} & 0 & \cdots & 0\end{bmatrix}$. Then, with equation (2.32) and take collocation points asequation(2.13), equation(2.34)is then becomes

$$\frac{t^\alpha}{\Gamma(\alpha+1)} = \mathbf{e}\mathbf{Q}^\alpha \mathbf{H}_m(t) \tag{2.35}$$

Expression in equation (2.35)is very helpful when to find inversion of Laplace transform later in irrational transfer function expression.

3. NUMERICAL ANALYSIS OF INVERSION LAPLACE TRANSFORM

The Laplace transform of a function $x(t)$, denoted by $X(s)$ is defined by an integral function equation

$$X(s) = \mathcal{L}\{x(t)\} = \int_0^\infty e^{-st}x(t)dt \tag{3.1}$$

We know the Laplace transform of integral is as below

$$\mathcal{L}\left\{\int_0^t \cdots \int_0^{t_3}\int_0^{t_2} x(\tau_1)\,d\tau_1\,d\tau_2 \ldots d\tau_n\right\} = \frac{X(s)}{s^n} \tag{3.2}$$

The integration in equation (3.2) and equation (2.18) are corresponding to the multiplication of $1/s$ in s domain and Haar operational matrix for integration $\mathbf{Q}_m$ in t domain respectively. Thus we could replace the $1/s$ factor to the generalised Haar operational matrix, $\mathbf{Q}_m$.

Assuming that the irrational transfer function has a form of

$$X(s) = \frac{a_0 + \dfrac{a_1}{s} + \dfrac{a_2}{s^2} + \cdots + \dfrac{a_n}{s^n}}{s^{\alpha+1}\left(b_0 + \dfrac{b_1}{s} + \dfrac{b_2}{s^2} + \cdots + \dfrac{b_n}{s^n}\right)} \tag{3.3}$$

where $0 \le \alpha < 1$ and truncated to $n\,(n \in \mathbb{N})$. By cross multiplying equation (3.3), we have

$$\left(b_0 + \frac{b_1}{s} + \cdots + \frac{b_n}{s^n}\right)X(s) = \frac{1}{s^{\alpha+1}}\left(a_0 + \frac{a_1}{s} + \cdots + \frac{a_n}{s^n}\right) \tag{3.4}$$

Then perform inverse Laplace transform of equation(3.4), at both side yields

$$\begin{aligned}&b_0 x(t) + b_1\int_0^t x(\tau)d\tau + \ldots + b_n\int_0^t \cdots \int_0^{t_3}\int_0^{t_2} x(\tau_1)\,d\tau_1\,d\tau_2 \ldots d\tau_n = \\ &= \frac{a_0 t^\alpha}{\Gamma(\alpha+1)} + \frac{a_1 t^{\alpha+1}}{\Gamma(\alpha+2)} + \cdots + \frac{a_n t^{\alpha+n}}{\Gamma(\alpha+n+1)}\end{aligned} \tag{3.5}$$

Taking the collocation points as equation (2.13), factorize $\mathbf{c}^T_m$, $\mathbf{e}$ and $\mathbf{H}_m$, equation (3.5) become

$$\begin{aligned}&\mathbf{c}^T_m(b_0\mathbf{I}_m + b_1\mathbf{Q}_m + b_2\mathbf{Q}^2_m + \cdots + b_n\mathbf{Q}^n_m)\mathbf{H}_m = \\ &= \mathbf{e}\mathbf{Q}^\alpha_m(a_0\mathbf{I}_m + a_1\mathbf{Q}_m + a_2\mathbf{Q}^2_m + \cdots + a_n\mathbf{Q}^n_m)\mathbf{H}_m\end{aligned} \tag{3.6}$$

Rewrite equation (3.6) with

$$\begin{aligned}g_1(\mathbf{Q}_m) &= a_0\mathbf{I}_m + a_1\mathbf{Q}_m + a_2\mathbf{Q}^2_m + \cdots + a_n\mathbf{Q}^n_m \\ g_2(\mathbf{Q}_m) &= b_0\mathbf{I}_m + b_1\mathbf{Q}_m + b_2\mathbf{Q}^2_m + \cdots + b_n\mathbf{Q}^n_m\end{aligned} \tag{3.7}$$

it becomes

$$\mathbf{c}_m^T g_2(\mathbf{Q}_m) = \mathbf{e}\mathbf{Q}_m^{\alpha} g_1(\mathbf{Q}_m) \quad (3.8)$$

Multiplying both sides with $g_2^{-1}(\mathbf{Q}_m)$, the vector coefficient $\mathbf{c}_m^T$ becomes

$$\mathbf{c}_m^T = \mathbf{e}\mathbf{Q}_m^{\alpha} g_1(\mathbf{Q}_m) g_2(\mathbf{Q}_m)^{-1} \quad (3.9)$$

Thus the inversion of Laplace transform is given by

$$\begin{aligned}
\mathbf{x} &= \mathbf{c}_m^T \mathbf{H}_m \\
&= \mathbf{e}\mathbf{Q}^{\alpha} g_1(\mathbf{Q}_m) g_2(\mathbf{Q}_m)^{-1}\mathbf{H}_m \\
&= \mathbf{e}\mathbf{Q}_m^{-1}\mathbf{H}_m\mathbf{H}_m^T\mathbf{Q}_m^{\alpha+1} g_1(\mathbf{Q}_m) g_2(\mathbf{Q}_m)^{-1}\mathbf{H}_m \\
&= \mathbf{e}\mathbf{Q}_m^{-1}\mathbf{H}_m\mathbf{H}_m^T \breve{\mathbf{X}}(\mathbf{Q}_m)\mathbf{H}_m \quad (3.10) \\
&= \mathbf{e}\mathbf{H}_m\mathbf{F}_{1m}^{-1}\mathbf{H}_m^T \breve{\mathbf{X}}(\mathbf{Q}_m)\mathbf{H}_m \\
&= \begin{bmatrix} \frac{2m}{\tau} & -\frac{2m}{\tau} & \cdots & -\frac{2m}{\tau} \end{bmatrix}_{1\times m} \mathbf{H}_m^T \breve{\mathbf{X}}(\mathbf{Q}_m)\mathbf{H}_m
\end{aligned}$$

where $\breve{\mathbf{X}}(\mathbf{Q}_m)$ is from equation (3.3) by replacing $1/s$ with the generalised Haar operational matrix, $\mathbf{Q}_m$.

4. NUMERICAL RESULTS OF INVERSION LAPLACE TRANSFORM

4.1 Example 1

Consider the irrational transfer function as

$$X(s) = \frac{1}{\sqrt{s^2+1}} \quad (3.11)$$

By using this method, firstly, find expression of $X(s)$ in terms of $1/s$ and denoted as $\hat{X}(1/s)$ as below

$$\hat{X}\left(\frac{1}{s}\right) = \frac{1/s}{\sqrt{1+(1/s)^2}} \quad (3.12)$$

Then, replace each terms of $1/s$ in equation (3.12) by the generalised Haar operational matrix $\mathbf{Q}_m$

$$\breve{\mathbf{X}}(\mathbf{Q}_m) = \mathbf{Q}_m(\mathbf{I}+\mathbf{Q}_m{}^2)^{(1/2)} \quad (3.13)$$

Lastly, by equation (3.10) the inversion of Laplace transform can be calculated by the below equation.

$$\mathbf{x} = \begin{bmatrix} \frac{2m}{\tau} & -\frac{2m}{\tau} & \cdots & -\frac{2m}{\tau} \end{bmatrix} \mathbf{H}_m^T\mathbf{Q}_m(\mathbf{I}+\mathbf{Q}_m{}^2)^{(1/2)}\mathbf{H}_m \quad (3.14)$$

In the case of $m=16$ and $\tau=4$, the result is shown in Figure 2.

4.2 Example 2

Consider the irrational transfer function as

$$X(s) = \frac{e^{-a\sqrt{s}}}{\sqrt{s}} \quad (3.15)$$

Expressing equation in terms of $1/s$, we obtain

$$\hat{X}\left(\frac{1}{s}\right) = \left(\frac{1}{s}\right)^{1/2} e^{-a(1/s)^{-1/2}} \quad (3.16)$$

Replace each $1/s$ with generalised Haar wavelet operational matrix, $\mathbf{Q}_m$.

$$\breve{\mathbf{X}}(\mathbf{Q}_m) = (\mathbf{Q}_m)^{1/2} e^{-a(\mathbf{Q}_m)^{-1/2}} \quad (3.17)$$

In the case of, $a=1$, $m=16$ and $\tau=1$, from equation (3.10), we obtain

$$\mathbf{x} = \begin{bmatrix} \frac{32}{1} & -\frac{32}{1} & \cdots & -\frac{32}{1} \end{bmatrix} \mathbf{H}_m^T(\mathbf{Q}_m)^{1/2} e^{-a(\mathbf{Q}_m)^{-1/2}}\mathbf{H}_m \quad (3.18)$$

The exact solution is

$$x(t) = \frac{e^{-a^2/4t}}{\sqrt{\pi t}} \quad (3.19)$$

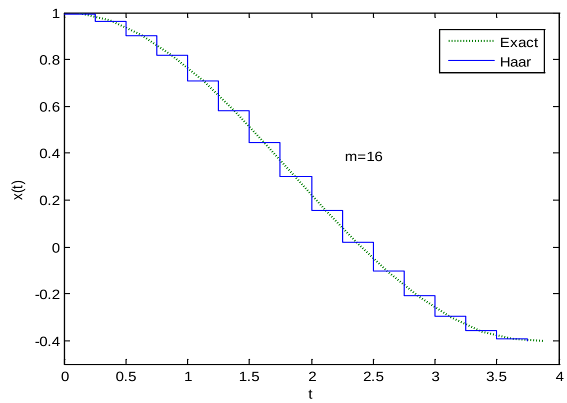


Fig. 2 Comparison between the exact solution and present numerical results for $m=16$

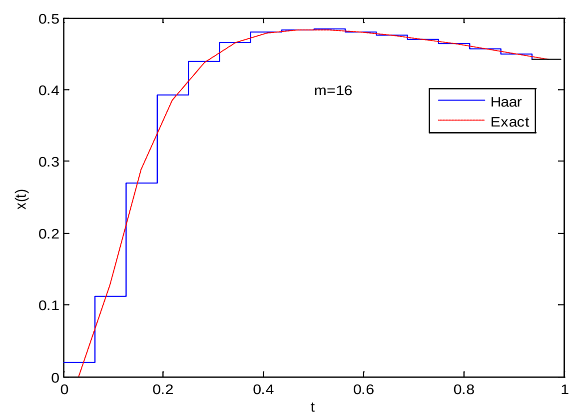


Fig. 3 Comparison between the exact solution and present numerical results for $m=16$

4.3 Example 3

Consider the irrational transfer function as

$$X(s) = \frac{a}{2\sqrt{\pi s^{3/2}}} e^{-a^2/4s} \quad (3.20)$$

Expressing equation in terms of $1/s$, we obtain

$$X\left(\frac{1}{s}\right)=\frac{a}{2\sqrt{\pi}}\left(\frac{1}{s}\right)^{3/2} e^{-a^2/4s} \tag{3.21}$$

Replace each $1/s$ with generalised Haar wavelet operational matrix, $\mathbf{Q}_m$.

$$\tilde{\mathbf{X}}(\mathbf{Q}_m)=\frac{a}{2\sqrt{\pi}}(\mathbf{Q}_m)^{3/2}\exp((-a^2/4)\mathbf{Q}_m) \tag{3.22}$$

In the case of, $a=1$, $m=32$ and $\tau=5$, from equation (3.10), we obtain

$$\mathbf{x}=\begin{bmatrix}\frac{64}{5} & -\frac{64}{5} & \cdots & -\frac{64}{5}\end{bmatrix}\mathbf{H}_{32}^{T}\frac{1}{2\sqrt{\pi}}(\mathbf{Q}_{32})^{\frac{3}{2}}\exp\left(\left(\frac{-1}{4}\right)\mathbf{Q}_{32}\right)\mathbf{H}_{32} \tag{3.23}$$

The exact solution is

$$x(t)=\frac{\sin a\sqrt{t}}{\pi} \tag{3.24}$$

and the result is shown in Figure 4

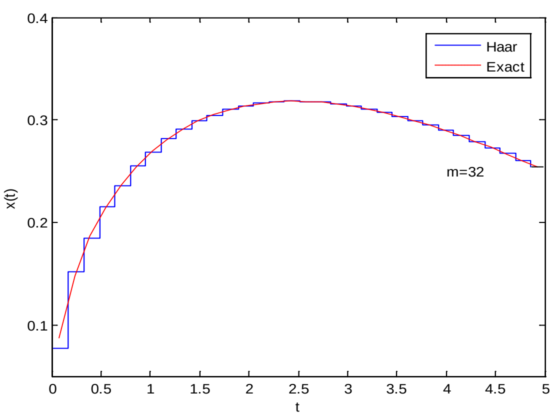


Fig. 4 Comparison between the exact solution and present numerical results for $m=32$

4.4 Example 4

Consider the irrational transfer function as

$$X(s)=\frac{e^{(1/s)}}{s\sqrt{s}} \tag{3.25}$$

Expressing equation in terms of $1/s$, we obtain

$$X\left(\frac{1}{s}\right)=\frac{1}{s}\left(\frac{1}{s}\right)^{1/2} \mathrm{e}^{1/s} \tag{3.26}$$

Replace each $1/s$ with generalised Haar wavelet operational matrix, $\mathbf{Q}_m$.

$$\tilde{\mathbf{X}}(\mathbf{Q}_m)=\mathbf{Q}_m(\mathbf{Q}_m)^{1/2}\mathrm{e}^{\mathbf{Q}_m} \tag{3.27}$$

In the case of $m=16$ and $\tau=10$, from equation (3.10), we obtain

$$\begin{aligned}\mathbf{x}&=\begin{bmatrix}\frac{2m}{\tau} & -\frac{2m}{\tau} & \cdots & -\frac{2m}{\tau}\end{bmatrix}\mathbf{H}_m^T\tilde{\mathbf{X}}(\mathbf{Q}_m)\mathbf{H}_m\\ &=\begin{bmatrix}\frac{32}{10} & -\frac{32}{10} & \cdots & -\frac{32}{10}\end{bmatrix}\mathbf{H}_m^T\mathbf{Q}_m(\mathbf{Q}_m)^{1/2}e^{\mathbf{Q}_m}\mathbf{H}_m\end{aligned} \tag{3.28}$$

The exact solution is

$$\frac{\sinh 2\sqrt{t}}{\sqrt{\pi}} \tag{3.29}$$

and the result is shown in Figure 5.

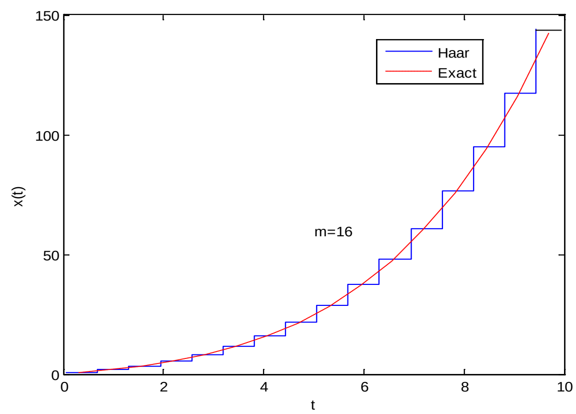


Fig. 5 Comparison between the exact solution and present numerical results for $m=32$

4.5 Example 5 – Initial Value Problem

Consider initial value problem as below

$$t\frac{d^2x}{dt^2}+\frac{dx}{dt}+tx=0 \tag{3.30}$$

The initial conditions for this case are given by

$$x(0)=1,\quad x'(0)=0 \tag{3.31}$$

The analytical solution for equation (3.30) is the Bessel function of zeroth kind, $J_0(t)$. The Laplace transform of equation (3.30) with respect to t is

$$-\frac{d}{ds}\left\{s^2X-sx(0)-x'(0)\right\}+sX-x(0)-\frac{dX}{ds}=0 \tag{3.32}$$

Then

$$(s^2+1)\frac{dX}{ds}+sX=0 \tag{3.33}$$

and integrating

$$X(s)=\frac{1}{\sqrt{s^2+1}} \tag{3.34}$$

The Laplace inversion for equation (3.34) is same as shown in Example 1.

4. CONCLUSION

In this paper the usage of the generalised Haar operational matrix into a unified method for finding the operational matrix of Haar has been enabled the method to find the inverse of Laplace transform for the whole domain of time. A few example of numerical inversion has been analysed and it is found that the present method shows a good agreement with analytical solution even for a small m. This method does not involve conventional and complex integration but only a few of sparse matrices manipulation. This method is considerable simple compared to conventional method and can be easily coded.

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