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High-order approximate solutions of strongly nonlinear cubic-quintic Duffing oscillator based on the harmonic balance method

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ABSTRACT

In this paper, a new reliable analytical technique has been introduced based on the Harmonic Balance Method (HBM) to determine higher-order approximate solutions of the strongly nonlinear cubic-quintic Duffing oscillator. The application of the HBM leads to very complicated sets of nonlinear algebraic equations. In this technique, the high-order nonlinear algebraic equations are approximated in the form of a power series solution, and this solution produces desired results even for small as well as large amplitudes of oscillation. Moreover, a suitable truncation formula is found in which the solution measures better results than existing results and it saves a lot of calculation. It is highly noteworthy that using the proposed technique, the third-order approximate solutions gives an excellent agreement as compared with the numerical solutions (considered to be exact). The proposed technique is applied to the strongly nonlinear cubic-quintic Duffing oscillator to reveals its novelty, reliability and wider applicability.

Introduction

Nonlinear oscillation in engineering and applied mathematics has been directed towards a topic to intensive research for many years [1–2]. The few issues occurring in different field of applied sciences and engineering are linear whereas a large number of oscillation problems are nonlinear. Nonlinear oscillations are important fact in physical sciences, mechanical structures and other discipline which are mathematically in the form of differential equations. Practically, all differential equations involving engineering and physical phenomena are nonlinear. The methods of solution techniques of the linear differential equations are comparatively easy and well established. On the contrary, the solution techniques of the Nonlinear Differential Equations (NDEs) are less available still now and, in general, linear approximations are frequently used. With the unearthing of numerous phenomena of the strongly nonlinear oscillation problems and in many cases of nonlinear mechanical vibrations of special types, the methods of large oscillations become insufficient for their analytical treatment.

In present decade, the nonlinear processes are one of the biggest challenges and not easy to control because the nonlinear characteristic of the system abruptly changes due to slight changes of valid parameters including time. As most of the physical and engineering phenomena in our world are essentially nonlinear and are described by NDEs. Obtaining exact solutions for these problems have many difficulties. It is very difficult to solve the NDEs and in general it is often more difficult to get an analytic approximation than a numerical one. To overcoming the shortcoming, in recent earlier, there are many analytical and numerical methods have been used for solving these NDEs. One of the most commonly used analytical technique for nonlinear oscillation systems is the Perturbation Methods [3–4], which involving expansion over a small parameter. However, most often there exist many nonlinear problems in applied science and engineering where small parameters do not exist and even if such small parameters do exist, the analytical solutions given by the perturbation methods involving expansion over a small parameter have a small range of validity. As the cubic–quintic Duffing oscillators considered in this paper include

Abbreviations: HEBM, He's Energy Balance Method; IPM, Iterative Perturbation Method; HOH, Higher-order Harmonic Terms; NDEs, Nonlinear Differential Equations; HBM, Harmonic Balance Method.

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Nomenclature

a_0	Initial Oscillation Amplitude
x	Dimensionless Displacement
u, v, w, z	Constant Coefficients
G_1, G_2	General Nonlinear Functions
k_1, k_2, k_3	Constant Parameters
$f(x)$	General Nonlinear Function
F_1, F_3, F_5	General Nonlinear Operators
t	Time
U_1, U_2, V_1, V_2	Arbitrary Constants
$Er(\%)$	Relative Error

Greek Symbols

π	Pi
α, β, γ	Constant Coefficients
ω	Angular Frequency
λ_0, μ_0	Small Quantities
ω_2, ω_3	Second- and Third-order Approximate Angular Frequency
ω_{ex}	Exact Frequency
ε	Constant Coefficient Parameter
ω_0	Constant Coefficient Parameter
ρ	Arbitrary Constant
ω_{trun2}, ω_{trun3}	Second- and Third-order Approximate Angular Frequency (Using Truncation Principle)

small as well as large parameters, the perturbation method involving expansion over a small parameter is not suitable.

These days, researchers have been used some other analytical methods (non-perturb) for the various strongly nonlinear oscillation systems to determine approximate periodic solutions, including, the Max-Min Approach [5–6], Approximate Method [7], Rational Variational Approaches [8], Amplitude Frequency Formulation [9], Global Error Minimization Method [10], Global Residue Rational Harmonic Balance Method [11], Energy Balance Method [12–13] and so forth. Most of the aforementioned methods consider only the lower-order approximate solutions which lead to low accuracy while it is usually difficult to achieve higher-order analytical approximations.

The cubic–quintic Duffing equation is found in the modeling of the free vibration of a restrained uniform beam carrying an intermediate lumped mass and undergoing large amplitudes of oscillation in the unimodel Duffing type temporal problem [19]. A differential equation having fifth power nonlinearity is very difficult to handle because of the presence of strong nonlinearity. Due to the presence of fifth power nonlinearity, the accuracy of approximate analytical methods becomes extremely demanding. In recent past, many researchers have been obtained approximate periodic solutions of the strongly nonlinear oscillation of cubic–quintic Duffing oscillator. Among them, Lai et al. [14] has been determined up to third-order approximations by using newton-harmonic balancing method. An iterative homotopy harmonic balance method has been applied to drive approximate periodic solutions of cubic–quintic Duffing oscillator by Guo et al. [15]. Using a method combining the features of the homotopy concept with the variational approach, Khan et al. [16] obtained up to fourth-order approximations. Ganji et al. [17] and Ganji et al. [18] has investigated approximate solutions applying He's energy balance method and iteration perturbation method respectively. Moreover, the improved energy balance method and the global residue harmonic balance method have been applied to require approximate periodic solutions of the strongly nonlinear oscillation of cubic–quintic Duffing oscillator by Akbarzade and Farshidianfar [19]. Pirbodaghi et al. [20] has explored the homotopy analysis method and homotopy pade technique to obtain approximate periodic solutions for the Duffing equation with cubic and quintic nonlinearities. Recently, Razzak [21] has considered an analytical method combined of the homotopy perturbation method and variational approach to acquire approximate periodic solutions. Very recently, Zuniga [22], Zakeri [23] and Beléndez et al. [24–25] approximated exact solution of the cubic–quintic Duffing oscillator. However, the solution procedure is complicated and the approximated solution contains a set of complex nonlinear algebraic equations with Jacobian elliptic functions which are not easily solved. In this situation, a suitable and efficient analytical approximate technique for strongly nonlinear oscillation system of cubic–quintic Duffing oscillator must be needed.

The harmonic balance method (HBM) [26–32] is another most efficient technique for solving strongly nonlinear oscillation systems where the nonlinear term is not small. Usually, a set of complicated nonlinear algebraic equations is appeared when the HBM is formulated. In the proposed technique, using a small parameter, the power series solutions yield desired results of these complicated nonlinear algebraic equations even for small as well as large oscillation amplitudes. Besides, a new suitable truncation principle of these nonlinear algebraic equations takes the solutions better than existing solutions and it saves a lot of calculations. The higher-order approximations (mainly third-order approximation) have been obtained of the strongly nonlinear oscillation system of cubic-quintic Duffing oscillators. Comparison of the approximate frequencies obtained in this paper with previously existing results and the corresponding exact frequencies which show that the proposed method gives the high accuracy results. The main advantage of this technique is that it provides a suitable way to calculate a set of complicated nonlinear algebraic equations and increases the rate of accuracy of approximate solutions as well as frequencies of the several benchmark strongly nonlinear oscillator problems arising in nonlinear sciences and engineering.

Solution approach

Let us consider a second-order nonlinear differential equation, can be expressed as

$$\ddot{x} + \omega_0^2 x = -\varepsilon f(x) \text{ and initial condition } [x(0) = a_0,\ \dot{x}(0) = 0], \tag{1}$$

where $f(x)$ is a nonlinear function such that $f(-x) = -f(x)$, $\omega_0 \geqslant 0$ and ε is a constant.

Consider a periodic solution of Eq. (1), can be considered as in the form

$$x = a_0(\rho\cos(\omega t) + u\cos(3\omega t) + v\cos(5\omega t) + w\cos(7\omega t) + z \times \cos(9\omega t)\ldots), \tag{2}$$

where a_0, ρ and ω are constants. If $\rho = 1 - u - v - \cdots$, then the solution Eq. (2) easily satisfies the initial conditions given in Eq. (1).

Substituting Eq. (2) into Eq. (1) and expanding $f(x)$ in a Fourier series, it becomes to an algebraic identity as

$$a_0[\rho(\omega_0^2-\omega^2)\cos(\omega t)+u(\omega_0^2-9\omega^2)\cos(3\omega t)+\cdots] = -\varepsilon[F_1(a_0,u,\ldots)\cos(\omega t)+F_3(a_0,u,\ldots)\cos(3\omega t)+\cdots] \quad (3)$$

By comparing the coefficients of equal harmonics of Eq. (3), the following nonlinear algebraic equations, can be shown as

$$\rho(\omega_0^2-\omega^2)=-\varepsilon F_1,\quad u(\omega_0^2-9\omega^2)=-\varepsilon F_3,\quad v(\omega_0^2-25\omega^2) = -\varepsilon F_5,\ldots \quad (4)$$

With help of the first equation, ω^2 is eliminated from all the remaining equations of Eq. (4). Thus, Eq. (4), can be transformed into the following form as

$$\rho\omega^2=\rho\omega_0^2+\varepsilon F_1,\quad 8\omega_0^2 u\rho=\varepsilon(\rho F_3-9uF_1),\quad 24\omega_0^2 v\rho = \varepsilon(\rho F_5-25vF_1),\ldots \quad (5)$$

Substitution $\rho=1-u-v-\cdots$, and simplification, second-, third-equations of Eq. (5), can be taken into the following form

$$u=G_1(\omega_0,\varepsilon,a_0,u,v,\ldots,\lambda_0),\quad v=G_2(\omega_0,\varepsilon,a_0,u,v,\ldots,\lambda_0),\ldots, \quad (6)$$

where $G_1,\ G_2,\ldots$ exclude respectively the linear terms of $u,\ v,\ldots$.

Whatever the values of ω_0, ε and a_0, there exists a parameter $\mu_0(\omega_0,\varepsilon,a_0)\ll 1$, such that $u,\ v,\ldots$ are expandable in the following power series in terms of λ_0 as

$$u=U_1\lambda_0+U_2\lambda_0^2+\cdots,\quad v=V_1\lambda_0+V_2\lambda_0^2+\cdots, \quad (7)$$

where $U_1,\ U_2,\ldots,V_1,\ V_2,\ldots$ are constants.

Finally, substituting the values of $u,\ v,\ldots$ from Eq. (7) into the first equation of Eq. (5), the angular frequency ω is determined. This completes the determination of all related unknows for the proposed periodic solution as given in Eq. (2).

General definition of cubic-quintic duffing oscillator

A cubic–quintic Duffing oscillator of a conservative autonomous system can be described by the following second-order differential equation with cubic–quintic nonlinearities which is stated in Lai et al. [14] as

$$\ddot{x}+f(x)=0, \quad (8)$$

with initial conditions of

$$x(0)=a_0,\quad \dot{x}(0)=0 \quad (9)$$

where $f(x)=\alpha x+\beta x^3+\gamma x^5$. x and t are generalized dimensionless displacement and time variable respectively and x is the function of t.

The exact frequency ω_{ex} is calculated in Lai et al. [14] by imposing the mentioned initial conditions is

$$\omega_{ex}(a_0)=\frac{\pi k_1}{2\int_0^{\pi/2}(1+k_2\sin^2 t+k_3\sin^4 t)^{-1/2}dt} \quad (10a)$$

$$k_1=\sqrt{\alpha+\frac{\beta a_0^2}{2}+\frac{\gamma a_0^4}{4}}, \quad (10b)$$

$$k_2=\frac{3\beta a_0^2+2\gamma a_0^4}{6\alpha+3\beta a_0^2+2\gamma a_0^4}, \quad (10c)$$

$$k_3=\frac{2\gamma a_0^4}{6\alpha+3\beta a_0^2+2\gamma a_0^4}. \quad (10d)$$

Application of HBM to cubic-quintic duffing oscillator

The complete solution procedure of cubic-quintic duffing oscillator (Eq. (8)) with $\alpha=\beta=\gamma=1$ is presented by a new reliable analytical technique based on the Harmonic Balance Method (HBM). Furthermore, the comparison of the obtained angular frequencies with previously existing results and the corresponding exact frequencies are listed in Table 1. This table corresponds the approximate angular frequencies for the cubic-quintic Duffing oscillators to small as well as large amplitude which obtained by the proposed method as described in the following example.

Let us consider $\alpha=\beta=\gamma=1$, into the Eq. (8), the general form of unforced cubic-quintic Duffing oscillator can be taken into the following form as

$$\ddot{x}+x+x^3+x^5=0. \quad (11)$$

Herein we have to determine second- and third-order approximations and frequencies for the strongly nonlinear cubic-quintic Duffing oscillators.

Let us consider a two-term solution, *i.e.*, $x=a_0(\rho\cos(\omega_2 t)+u\cos(3\omega_2 t))$ for the Eq. (11). Substituting this solution along with $\rho=1-u$ into the Eq. (11), the Eq. (3), can be transformed into

$$(1-u)\omega_2^2\cos(\omega_2 t)+9u\omega_2^2\cos(3\omega_2 t) = (1+3a_0^2/4+5a_0^4/8-u-3a_0^2u/2-25a_0^4u/16 +9a_0^2u^2/4+\cdots)\cos(\omega_2 t)+(a_0^2/4+5a_0^4/16+u +3a_0^2u/4-9a_0^2u^2/4+\cdots)\cos(3\omega_2 t)+HOH, \quad (12)$$

where HOH represents the higher-order harmonics term.

Now comparing the coefficients of equal harmonics, the following equations can be obtained as

$$(1-u)\omega_2^2 = 1+3a_0^2/4+5a_0^4/8-u-3a_0^2u/2-25a_0^4u/16 +9a_0^2u^2/4+15a_0^4u^2/4+\cdots 9u\omega_2^2 = a_0^2/4+5a_0^4/16+u+3a_0^2u/4+5a_0^4u/16 -9a_0^2u^2/4-5a_0^4u^2/2+\cdots \quad (13)$$

From the first equation of Eq. (13), it can easily be written into another form as

$$\omega_2^2=(1+3a_0^2/4+5a_0^4/8-u-3a_0^2u/2-25a_0^4u/16 +9a_0^2u^2/4+15a_0^4u^2/4+\cdots)/(1-u) \quad (14)$$

By elimination of ω_2^2 from the second equations of Eq. (13), with the help of Eq. (14) and simplification, the following nonlinear algebraic equation of u can be found as

$$-a_0^2/4-5a_0^4/16+8u+25a_0^2u/4+45a_0^4u/8-8u^2 -21a_0^2u^2/2-45a_0^4u^2/4+16a_0^2u^3+25a_0^4u^3-23a_0^2u^4/2 -675a_0^4u^4/16+355a_0^4u^5/8-85a_0^4u^6/4=0 \quad (15)$$

The Eq. (15), can be written into another from as

$$u=\lambda_0(4+5a_0^2+128u^2/a_0^2+168u^2-256u^3+184u^4 -710a_0^2u^5+\cdots), \quad (16)$$

where $\lambda_0=a_0^2/(128+100a_0^2+90a_0^4)$.

The power series solution of Eq. (16) in terms of λ_0, one could obtain

$$u = (4+5a_0^2)\lambda_0 + (7808 + 2048/a_0^2 + 12800a_0^2 + 11400a_0^4 + 4500a_0^6)\lambda_0^3 + \cdots, \tag{17}$$

Substituting the value of u from Eq. (17) into the Eq. (14) and then simplification, the second-order approximate angular frequency can be obtained as

$$\omega_2 = \sqrt{(16-36a_0^2-110a_0^4-75a_0^6)\lambda_0/16+(384a_0^2+1680a_0^4+2400a_0^6+1125a_0^8)\lambda_0^2/16+\cdots} \tag{18}$$

Thus, a two-term solution (i.e. second approximation) of Eq. (11) is

$$x = a_0\cos(\omega_2 t) + a_0 u(\cos(3\omega_2 t) - \cos(\omega_2 t)), \tag{19}$$

where u and ω_2 are given respectively by Eqs. (17) and (18).

It can clearly be observed that the solution Eq. (19) gives better results when the truncation principle is applied in Eq. (13). Therefore, after applying truncation principle in Eq. (13), it can be taken into the following form as

$$\begin{aligned}(1-u)\omega_2^2 = {} & 1 + 3a_0^2/4 + 5a_0^4/8 - u - 3a_0^2u/2 - 25a_0^4u/16 \\ & + 9a_0^2u^2/8 + 15a_0^4u^2/89u\omega_2^2 \\ = {} & a_0^2/4 + 5a_0^4/16 + u + 3a_0^2u/4 + 5a_0^4u/16 \\ & - 9a_0^2u^2/8 - 5a_0^4u^2/4\end{aligned} \tag{20}$$

In Eq. (20), it is clear that the higher-order terms of u (more than second) are ignored; but half of the second order terms are considered. Now, from the first equation of Eq. (20), one could have

$$\omega_2^2 = (1 + 3a_0^2/4 + 5a_0^4/8 - u - 3a_0^2u/2 - 25a_0^4u/16 + 9a_0^2u^2/8 + 15a_0^4u^2/8)/(1-u) \tag{21}$$

By elimination of ω_2^2 from the second equations of Eq. (20) with the help of Eq. (21) and then simplification, the following nonlinear algebraic equation of u can be determined as

$$-a_0^2/4 - 5a_0^4/16 + 8u + 25a_0^2u/4 + 45a_0^4u/8 - 8u^2 - 93a_0^2u^2/8 - 25a_0^4u^2/2 + 9a_0^2u^3 + 125a_0^4u^3/8 = 0 \tag{22}$$

The Eq. (22) can be written into another form as

$$u = \lambda_0(4 + 5a_0^2 + 128u^2/a_0^2 + 186u^2 + 200a_0^2u^2 - 144u^3 - 250a_0^2u^3), \tag{23}$$

where λ_0 is defined in Eq. (16).

The power series solution of Eq. (23) in terms of λ_0, one could get

$$u = (4+5a_0^2)\lambda_0 + (8096 + 2048/a_0^2 + 13840a_0^2 + 12650a_0^4 + 5000a_0^6)\lambda_0^3 + \cdots \tag{24}$$

Substituting the value of u from Eq. (24) into the Eq. (21) and then simplification, the second-order approximate angular frequency using truncation principle is

$$\omega_{trun2} = \sqrt{(16-36a_0^2-110a_0^4-75a_0^6)\lambda_0/16+(96a_0^2+480a_0^4+750a_0^6+375a_0^8)\lambda_0^2/16+\cdots} \tag{25}$$

In a similar way, the method can be used to determine higher-order approximations. In this article, a third-order approximate solution can be considered as

$$x(t) = a_0\cos(\omega_3 t) + a_0 u(\cos(3\omega_3 t) - \cos(\omega_3 t)) + a_0 v(\cos(5\omega_3 t) - \cos(\omega_3 t)) \tag{26}$$

Using Eq. (26) into the Eq. (11) and equating the coefficients of same harmonic terms $\cos(\omega_3 t)$, $\cos(3\omega_3 t)$ and $\cos(5\omega_3 t)$, the related equations can be shown as

$$\begin{aligned}(1-u-v)\omega_3^2 = {} & 1 + 3a_0^2/4 + 5a_0^4/8 - u - 3a_0^2u/2 - 25a_0^4u/16 \\ & + 9a_0^2u^2/4 + 15a_0^4u^2/4 - 3a_0^2u^3/2 - 25a_0^4u^3/4 \\ & + 25a_0^4u^4/4 - 45a_0^4u^5/16 - v - 9a_0^2v/4 \\ & - 45a_0^4v/16 + 9a_0^2uv/2 + 10a_0^4uv \\ & - 3a_0^2u^2v - 75a_0^4u^2v/4 + \cdots 9u\omega_3^2 \\ = {} & a_0^2/4 + 5a_0^4/16 + u + 3a_0^2u/4 \\ & + 5a_0^4u/16 - 9a_0^2u^2/4 - 5a_0^4u^2/2 + 2a_0^2u^3 \\ & + 25a_0^4u^3/4 - 125a_0^4u^4/16 \\ & + 65a_0^4u^5/16 - 5a_0^4v/16 - 3a_0^2uv/2 - 5a_0^4uv/2 \\ & + 3a_0^2u^2v/2 + 75a_0^4u^2v/8 + \cdots 25v\omega_3^2 = a_0^4/16 \\ & + 3a_0^2u/4 + 15a_0^4u/16 - 3a_0^2u^2/4 - 5a_0^4u^2/2 \\ & + 25a_0^4u^3/8 - 25a_0^4u^4/16 - a_0^4u^5/16 + v \\ & + 3a_0^2v/2 + 25a_0^4v/16 - 9a_0^2uv/2 - 35a_0^4uv/4 \\ & + 15a_0^2u^2v/4 + 75a_0^4u^2v/4 + \cdots\end{aligned} \tag{27}$$

From the first equation of Eq. (27), one could have

$$\begin{aligned}\omega_3^2 = {} & (1 + 3a_0^2/4 + 5a_0^4/8 - u - 3a_0^2u/2 - 25a_0^4u/16 \\ & + 9a_0^2u^2/4 + 15a_0^4u^2/4 - 3a_0^2u^3/2 - 25a_0^4u^3/4 \\ & + 25a_0^4u^4/4 - 45a_0^4u^5/16 - v - 9a_0^2v/4 - 45a_0^4v/16 \\ & + 9a_0^2uv/2 + 10a_0^4uv - 3a_0^2u^2v - 75a_0^4u^2v/4 + \cdots)/(1 \\ & - u - v)\end{aligned} \tag{28}$$

Eliminating ω_3^2 from last two equations of Eq. (27) with the help of Eq. (28), the simplified form of the nonlinear algebraic equations of u and v can be written as

$$u = \lambda_0(4 + 5a_0^2 + 168u^2 - 256u^3 - 4v - 10a_0^2v + 288uv + 128uv/a_0^2 - 564u^2v + \cdots) \tag{29}$$

$$v = \mu_0(a_0^2 + +12u + 14a_0^2u - 24u^2 + 12u^3 + 492uv + 384uv/a_0^2 - 756u^2v + 540u^3v + \cdots), \tag{30}$$

where λ_0 is defined in Eq. (16) and $\mu_0 = a_0^2/(384 + 276a_0^2 + 226a_0^4)$.

The algebraic relation between λ_0 and μ_0 is

$$\mu_0 = \frac{(64 + 50a_0^2 + 45a_0^4)\lambda_0}{192 + 138a_0^2 + 113a_0^4} \tag{31}$$

Therefore, Eq. (30) can be transformed into

$$v = \frac{(64 + 50a_0^2 + 45a_0^4)\lambda_0}{192 + 138a_0^2 + 113a_0^4}(a_0^2 + +12u + 14a_0^2u - 24u^2 + 12u^3 + 492uv - 756u^2v + \cdots). \tag{32}$$

The power series solutions of Eq. (29) and Eq. (32) are obtained in terms of λ_0 as

$$u = (4+5a_0^2)\lambda_0 - \frac{(256 + 840a_0^2 + 680a_0^4 + 450a_0^6)}{192 + 138a_0^2 + 113a_0^4}a_0^2\lambda_0^2 + \cdots, \tag{33}$$

$$v = \frac{(64+50a_0^2+45a_0^4)a_0^2\lambda_0 + (3072+9824a_0^2+12440a_0^4+8720a_0^6+3150a_0^8)\lambda_0^2}{192+138a_0^2+113a_0^4} + \cdots. \tag{34}$$

Now substituting the values of u and v from Eqs. (33) and (34) into Eq. (28), the third-order approximate angular frequency is

$$\omega_3=\sqrt{\left(-\frac{(-1536+2352a_0^2+12908a_0^4+18544a_0^6+12805a_0^8+5025a_0^{10})\lambda_0}{8(192+138a_0^2+113a_0^4)}+\cdots\right)} \tag{35}$$

Therefore, a third-order approximate periodic solution of Eq. (11) is defined in Eq. (26) where u, v and ω_3 are respectively given by the Eqs. (33)–(35).

Applying truncation principle, Eq. (27) can be transformed into

$$\begin{aligned}(1-u-v)\omega_3^2 &= 1+3a_0^2/4+5a_0^4/8-u-3a_0^2u/2\\ &\quad -25a_0^4u/16+9a_0^2u^2/4+15a_0^4u^2/4\\ &\quad -3a_0^2u^3/4-25a_0^4u^3/8-v-9a_0^2v/4\\ &\quad -45a_0^4v/16+9a_0^2uv/4+5a_0^4uv9u\omega_3^2\\ &= a_0^2/4+5a_0^4/16+u+3a_0^2u/4+5a_0^4u/16\\ &\quad -9a_0^2u^2/4-5a_0^4u^2/2+a_0^2u^3+25a_0^4u^3/8\\ &\quad -5a_0^4v/16-3a_0^2uv/4-5a_0^4uv/425v\omega_3^2\\ &= a_0^4/16+3a_0^2u/4+15a_0^4u/16-3a_0^2u^2/4\\ &\quad -5a_0^4u^2/2+25a_0^4u^3/16+v+3a_0^2v/2\\ &\quad +25a_0^4v/16-9a_0^2uv/4-35a_0^4uv/8\end{aligned} \tag{36}$$

From the first equation of Eq. (36), one could obtain

$$\begin{aligned}\omega_3^2 &= (1+3a_0^2/4+5a_0^4/8-u-3a_0^2u/2-25a_0^4u/16\\ &\quad +9a_0^2u^2/4+15a_0^4u^2/4-3a_0^2u^3/4-25a_0^4u^3/8-v-9a_0^2v/\\ &\quad 4-45a_0^4v/16+9a_0^2uv/4+5a_0^4uv)/(1-u-v)\end{aligned} \tag{37}$$

Removing ω_3^2 from last two equations of Eq. (36), with the help of Eq. (37) and then simplification, the nonlinear algebraic equations of u and v are

$$\begin{aligned}u &= \lambda_0(4+5a_0^2+168u^2-272u^3-4v-10a_0^2v+300uv\\ &\quad +128uv/a_0^2-276u^2v+\cdots)\end{aligned} \tag{38}$$

$$\begin{aligned}v &= \mu_0(a_0^2+12u+14a_0^2u-24u^2+12u^3+528uv\\ &\quad +384uv/a_0^2-852u^2v+300u^3v+\cdots),\end{aligned} \tag{39}$$

where λ_0 and μ_0 are defined in Eq. (16) and Eq. (30).

Therefore, the power series solutions of Eq. (38) and Eq. (39) in terms of λ_0 can be determined as

$$u=(4+5a_0^2)\lambda_0-\frac{(256+840a_0^2+680a_0^4+450a_0^6)}{192+138a_0^2+113a_0^4}a_0^2\lambda_0^2+\cdots, \tag{40}$$

$$v=\frac{(64+50a_0^2+45a_0^4)a_0^2\lambda_0+(3072+9824a_0^2+12440a_0^4+8720a_0^6+3150a_0^8)\lambda_0^2}{192+138a_0^2+113a_0^4}+\cdots. \tag{41}$$

Now substituting the values of u and v from Eqs. (40) and (41) into Eq. (37), the third-order approximate angular frequency using truncation principle is

$$\omega_{trun3}=\sqrt{\left(-\frac{(-1536+2352a_0^2+12908a_0^4+18544a_0^6+12805a_0^8+5025a_0^{10})\lambda_0}{8(192+138a_0^2+113a_0^4)}+\cdots\right)} \tag{42}$$

Therefore, a third-order approximation periodic solution (using truncation principle) of Eq. (11) is defined in Eq. (26) where u, v and ω_3 are respectively given by the Eqs. (40)–(42).

Results and discussions

We illustrate the accuracy of the second- and third- order approximate frequencies and their relative errors $Er(\%)$ obtained in this paper by using truncation and without truncation principle for the strongly nonlinear cubic-quintic Duffing oscillator which are listed in Table 1. It can clearly be seen that all the approximate frequencies obtained in this article applying the proposed technique are better than those obtained previously by Lai et al. [14], Ganji et al. [17] and Ganji et al. [18]. Moreover, the maximum relative error calculated in this article by using truncation principle (third-order approximation) is 0.04% whereas in the previously published articles Ganji et al. [17] and Ganji et al. [18] are 2.27% and 5.79% respectively. It has been mentioned that the solution procedures of Lai et al. [14], Ganji et al. [17] and Ganji et al. [18] are laborious specially to obtain higher-order approximations. High accuracy results and very simple solution procedure guaranteed us the proposed technique is much more efficient than existing several methods.

Table 1
Comparison the approximate frequencies with the existing results and the corresponding exact frequency Lai et al. ω_{ex} [14] for the cubic-quintic Duffing oscillator.

a_0	ω_2 Er (%)	ω_3 Er (%)	ω_{trun2} Er (%)	ω_{trun3} Er (%)	ω_{ex}	$\omega_{[17]}^{HEBM}$ Er (%)	$\omega_{[18]}^{IPM}$ Er (%)
0.10	1.003772	1.003772	1.003772	1.003772	1.003770	1.003773	1.003774
	0.0001	0.0001	0.0001	0.0001		0.0002	0.0003
0.30	1.035540	1.035538	1.035540	1.035538	1.035540	1.035492	1.035645
	0.0000	0.0001	0.0000	0.0001		0.0046	0.0101
0.50	1.106591	1.106545	1.106578	1.106544	1.106540	1.106356	1.107502
	0.0046	0.0004	0.0034	0.0003		0.0166	0.0869
1.0	1.526041	1.523762	1.525150	1.523633	1.523590	1.527720	1.541103
	0.1608	0.0112	0.1023	0.0028		0.2710	1.1494
3.0	7.367509	7.284161	7.327622	7.271511	7.268630	7.417768	7.640353
	1.3603	0.2136	0.8115	0.0396		2.0518	5.1140
5.0	19.482606	19.231700	19.362576	19.190477	19.181500	19.608793	20.257714
	1.5697	0.2617	0.9440	0.0468		2.2276	5.6106
8.0	49.086138	48.428975	48.772419	48.318259	48.294600	49.390624	51.078371
	1.6389	0.2782	0.9893	0.0489		2.2694	5.7641
10.0	76.421355	75.389512	75.929091	75.214696	75.177400	76.889585	79.536155
	1.6546	0.2821	0.9998	0.0496		2.2775	5.7979

Note: ω_{trun2} and ω_{trun3} respectively denote second- and third- order approximate frequencies obtained by truncation principle. ω_2 and ω_3 represent second- and third- order approximate frequencies where truncation principle is not used. $\omega_{[17]}^{HEBM}$ and $\omega_{[18]}^{IPM}$ indicate the approximate frequencies obtained previously in Ganji et al. [17] and Ganji et al. [18]. ω_{ex} represents the exact frequency which is stated in Lai et al. [14]. $Er(\%)$ denotes the percentage error obtained by the relation $|\frac{\omega_{ex}-\omega_i}{\omega_{ex}}\times 100|$ where $i=2,3$.

Conclusion

In this paper, a new reliable analytical technique based on the Harmonic Balance Method (HBM) has been established to drive approximate periodic solutions of the strongly nonlinear oscillations with cubic and quintic terms in the restoring force, the so-called cubic–quintic Duffing oscillators. The approximate frequencies show an excellent agreement comparing to exact frequency. And it is noted that the third-order approximate frequencies obtained by using truncation principle are almost similar with respect to the exact frequency. The most important advantages of the proposed method as compared to the previous methods are its simplicity, efficiency, flexibility in application, and avoidance of algebraic complexity.

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