Magnetohydrodynamic (MHD) Jeffrey fluid over a stretching vertical surface in a porous medium

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Abstract This paper presents the study of steady two-dimensional mixed convection boundary layer flow and heat transfer of a Jeffrey fluid over a stretched sheet immersed in a porous medium in the presence of a transverse magnetic field. The governing partial differential equations are reduced to nonlinear ordinary differential equations with the aid of similarity transformation, which are then solved numerically using an implicit finite difference scheme. The effects of some of the embedded parameters, such as Deborah number $\beta$, magnetic parameter $M$, mixed convection parameter $\lambda$, porosity parameter $\gamma$ and Prandtl number $Pr$, on the flow and heat transfer characteristics, are given in forms of tables and graphs.

1. Introduction

Flow of non-Newtonian fluid over stretching sheet has caught researchers’ attention in the last few decades due to its important practical applications, mainly in manufacturing and industry processes. For instance, in the extrusion of polymer process, the extrudate from the die is generally drawn and simultaneously stretched into sheet of desired thickness, and is then solidified. The final quality of the sheet depends mainly on the extensibility of the sheet and rate of heat transfer. Therefore, the cooling procedure has to be monitored adequately. To the best knowledge of the authors, the boundary layer flow over a moving horizontal sheet was first initiated by Sakiadis \cite{1}, who developed the flow field due to a flat surface. His work was later extended by Crane \cite{2} to a stretching sheet, for the two-dimensional...
problem where surface velocity is proportional to the distance from a fixed point. Since then, extensive research has been done capturing the various physical conditions and rheology of the fluids with different conditions, see for example Refs. [3–10].

Flow of an electrically-conducting fluid subject to a magnetic field has important applications, such as cooling nuclear reactors and magnetohydrodynamic (MHD) generators, plasma studies, oil exploration, geothermal energy extraction and boundary layer control in the field of aerodynamics [11]. In metallurgical processes, such as drawing, annealing and thinning of copper wires which involve cooling of continuous strips or filaments, the MHD effect is believed to improve the rate of cooling and hence, the properties of the final products. Mansur and Ishak [12] studied numerically magnetohydrodynamic (MHD) boundary layer flow of a nanofluid past a stretching/shrinking sheet with velocity, thermal, and solutal slip boundary conditions. Siddheshwar and Mahabaleshwar [13] examined analytically MHD flow of micropolar fluid over linear stretching sheet using regular perturbation technique and Ahmed et al. [14] applied the successive linearization method to study the effects of radiation and viscous dissipation on MHD boundary layer convective heat transfer with low pressure gradient in porous media. Other studies on the MHD flow in different fluids as well as different physical situations were considered for example in Refs. [15–22].

Due to its great range of applications in various fields, the investigation of convective heat transfer in fluid-saturated porous media has become a subject of interest, especially in geothermal energy recovery, food processing, fibre and granular insulation, design of packed bed reactors and dispersion of chemical contaminants in various processes in the chemical industry and environment [23]. Comprehensive studies can be found in Vafai [24], Nield and Bejan [25] and Vadasz [26]. There is an abundance of literature available which discusses fluid flow over stretching surfaces in porous medium. Some of them are Gbadeyan et al. [27] who investigated the effects of thermal diffusion and diffusion thermos effects on combined heat and mass transfer on mixed convection boundary layer flow over a stretching vertical sheet in a porous medium filled with a viscoelastic fluid in the presence of magnetic field, Imran et al. [28] studied the analysis of an unsteady mixed convection flow of a fluid saturated porous medium adjacent to heated/cooled semi-infinite stretching vertical sheet in the presence of heat source and Aly and Ebaid [29] investigated the mixed convection boundary-layer nano-fluids flow along an inclined plate embedded in a porous medium using both analytical and numerical approaches. Dessie and Kishan [30] examined the MHD boundary layer flow and heat transfer of a fluid with variable viscosity through a porous medium towards a stretching sheet along with viscous dissipation and heat source/sink effects. Narayana [31] carried out a study on the effects of radiation and first-order chemical reaction on unsteady mixed convection flow of a viscous incompressible electrically conducting fluid through a porous medium of variable permeability between two long vertical non conducting wavy channels in the presence of heat generation, and to name a few.

Jeffrey fluid is a type of non-Newtonian fluid that uses a relatively simpler linear model using time derivatives
instead of convected derivatives, which are used by most fluid models. Recently, this model of fluid has prompted active discussion. Some of the studies can be found in Shehzad et al. [32], Nallapu and Radhakrishnamacharya [33], Ahmad and Ishak [34] and Prasad et al. [35]. In view of the above discussions, the aim of this paper is to investigate the effects of MHD Jeffrey fluid flow embedded in porous medium over vertical stretching sheet. The model of the Jeffrey fluid flow is presented mathematically and has been solved numerically using a finite difference scheme.

2. Analysis

Consider the unsteady two-dimensional incompressible Jeffrey fluid in a porous medium over a vertical stretching sheet coinciding with the plane \( y=0 \), with the flow being confined to \( y>0 \). The surface is assumed to stretch with velocity \( u_0 = ax \), where \( a \) is stretching constant. Here, the \( x \)-axis is chosen parallel to the vertical surface and the \( y \)-axis is taken normal to it. The plate temperature is \( T_w = T_w(y) \), where \( T_w \) is the surface temperature, \( T_{\infty} \) is the ambient fluid temperature and \( b \) is constant. \( T_{\infty} > T_w \) and \( T_w < T_{\infty} \) are for heated surface (assisting flow) and cooled surface (opposing flow), respectively. A uniform transverse magnetic field of strength \( B_0 \) is applied parallel to the \( y \)-axis. By invoking the boundary layer and Boussinesq approximations, the governing boundary layer equations for this problem can be written as

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}
\]

\[
\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{\nu}{1 + \lambda_1} \left[ \frac{\partial^2 u}{\partial y^2} + \lambda_2 \left( \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} + \frac{\partial v}{\partial y} \frac{\partial u}{\partial y} - \frac{\partial^2 u}{\partial y^2} \right) \right]
+ g \beta T (T-T_{\infty}) \frac{\nu}{\rho} \frac{\sigma B_0^2 \gamma u}{\rho} - \frac{\gamma u}{\nu} \frac{\sigma B_0^2 \gamma u}{\rho} \tag{2}
\]

\[
\frac{\partial T}{\partial x} + \frac{\partial T}{\partial y} = \frac{\alpha^2 T}{\partial y^2} \tag{3}
\]

subject to the boundary conditions

\[
\begin{align*}
& u = u_0, \quad v = 0, \quad T = T_w \quad \text{at} \quad y = 0 \\
& u \to 0, \quad \frac{\partial T}{\partial y} \to 0, \quad T \to T_{\infty} \quad \text{as} \quad y \to \infty
\end{align*} \tag{4}
\]

where \( u \) and \( v \) are the velocity components in the \( x \) and \( y \) directions, respectively, \( \lambda_1 \) is the ratio of the relaxation and retardation times; \( \lambda_2 \) is the relaxation time and \( T \) is the fluid temperature. \( \nu = \frac{\mu}{\rho} \) is the kinematic viscosity, where \( \mu \) is the coefficient of fluid viscosity and \( \rho \) is the fluid density. \( g, \beta, \gamma, \epsilon, \sigma \) are gravitational acceleration, thermal expansion coefficient, permeability coefficient of porous medium and fluid electrical conductivity, respectively. Setting,

\[
\eta = \sqrt{\frac{\mu}{\gamma}}, \quad \psi = -\sqrt{\mu x f(\eta)}, \quad \theta = \frac{T-T_{\infty}}{T_w-T_{\infty}}, \tag{5}
\]

and making use of \( u = \partial \psi / \partial y \) and \( v = -\partial \psi / \partial x \), Eq. (1) is automatically satisfied and Eqs. (2), (3) reduced to

\[
f'' + \beta (f^2 - f'') + (1 + \lambda_1) \times \left[ ff'' - f' (\gamma + M) + \lambda \theta \right] = 0, \tag{6}
\]

\[
\theta'' + Pr (f\theta' - f') = 0, \tag{7}
\]

and the transformed boundary conditions can be written as

\[
f(0) = 0, \quad f'(0) = 1, \quad \theta(0) = 1 \quad \text{at} \quad \eta = 0 \\
\]

\[
f'(\eta) \to 0, \quad f''(\eta) \to 0, \quad \theta(\eta) \to 0 \quad \text{as} \quad \eta \to \infty \tag{8}
\]

where \( f \) is the dimensionless stream function, \( \theta \) is the dimensionless temperature and the prime denotes differentiation with respect to \( \eta \). Here, \( \beta \) is the Deborah number, \( \gamma \) is the porosity parameter, \( M \) is the MHD parameter, \( Pr \) is the Prandtl number and \( \lambda \) is the mixed convection parameter, which is defined as

\[
\beta = a\lambda_2, \quad \gamma = \frac{\nu}{\nu_{\infty}}, \quad M = \frac{\sigma B_0^2}{\rho a}, \quad Pr = \frac{\nu}{\alpha}, \quad \lambda = \frac{Gr}{Re^2} \tag{9}
\]

where \( Gr = g\beta (T_w-T_{\infty}) \gamma^3 \nu^2 \) and \( Re = u_0 \nu / \nu \) are the local Grashoff number and the local Reynolds number, respectively. It should be pointed out that \( \lambda > 0 \) and \( \lambda < 0 \) represent assisting flow (heated plate) and opposing flow (cooled plate), respectively, while \( \lambda = 0 \) corresponds to forced convection regime and \( \lambda \) corresponds to the free convection regime.

It is worth mentioning that when \( \lambda_1 = \beta = 0 \), Eqs. (6), (7) reduce to those of Gbadeyan et al. [27] when \( \kappa = \kappa - Du = Le = 0 \), as in their paper. The important physical quantities of interest are the skin friction coefficient \( C_f \) and the local Nusselt number \( Nu_s \), the transformed forms of which are given by Shehzad et al. [32], i.e

\[
C_f Re_s^{1/2} = \frac{1 + \beta}{1 + \lambda_1} f''(0), \quad Nu_s Re_s^{-1/2} = -\theta'(0), \tag{10}
\]

where \( Re_s = u_0 x / \nu \) is the local Reynolds number.

3. Results and discussion

Eqs. (7), (8), subject to boundary conditions (9), have been solved numerically using the finite-difference method, namely the Keller-box method for some arbitrary values of the Deborah number \( \beta \), the porosity parameter \( \gamma \), the MHD parameter \( M \), the mixed convection parameter \( \lambda \) and the Prandtl number \( Pr \), with the ratio of the relaxation and retardation times \( \lambda_1 \) held fixed (\( = 0 \)). To validate the accuracy of the numerical code used, the results obtained
agreement, as tabulated in Table 1.

Results in () and [ ] are those of Ishak et al. [36] and Imran et al. [28], respectively.

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are compared with previously published results by Ishak et al. [36] and Imran et al. [28], and found to be in good agreement, as tabulated in Table 1.

The numerical output obtained for the surface shear stress $f^*(0)$ and the local heat transfer $\theta^*(0)$ when $\beta=0$ (Newtonian fluid) and $\beta=0.5$ (Jeffrey fluid) are presented in Table 2 for several values of $Pr$ and $\gamma$, taking into account $M=1$ and $M=10$, respectively. The effects of $Pr$ number is observed to decrease $f^*(0)$ and increase $\theta^*(0)$. As $Pr$ increases, the thermal diffusivity decreases and thus, the heat is diffused away from the heated surface slowly, which results in higher heat transfer at the surface.

Full pictures of the effect of $\gamma$ towards the surface shear stress $f^*(0)$ and the local Nusselt number $\theta^*(0)$ when $\beta=0$, 0.5 and 2, $Pr=0.7$, $\lambda=1$ with $M=1$ and 3 are depicted in Figures 1 and 2, respectively. Both Table 2 and Figures 1 and 2 conclude that an $M$ increment will lead to both decrement of surface shear stress $f^*(0)$ and the local Nusselt number $\theta^*(0)$. Supplementary evidence is found in Figures 3 and 4 at fixed value of $\lambda$ and $\gamma$. Further, it is noted from Figures 1 and 2 that the surface shear stress $f^*(0)$ and the local Nusselt number $\theta^*(0)$ are also found to decrease with the increment of $\gamma$ for fixed value of $\beta$, $M$, $Pr$ and $\lambda$. These behaviours are consistent with the results plotted in Figures 3 and 4. The effect of the mixed convection parameter $\lambda$ is seen to increase both $f^*(0)$ and $\theta^*(0)$, with the increment of $\lambda$ as depicted in Figures 3 and 4. This is because the existence of the buoyancy force
induces a favourable pressure gradient that enhances the flow (increases the velocity $f'(\eta)$) and heat transfer in the boundary layer. This is in line with the velocity profile $f'(\eta)$ plotted in Figure 9, which is evidenced in the behaviour of the fluid motion.

The resulting profiles of the dimensionless velocity $f'(\eta)$ and the temperature distribution $\theta(\eta)$ for various values of the Deborah number $\beta$ and $\gamma$ when $Pr=0.7$, $\lambda=1$ and $M=1$, are displayed in Figures 5 and 6, respectively. It is observed that the velocity and boundary layer thickness are increasing functions of the Deborah number $\beta$. It should be pointed out that $\beta=0$ represents Newtonian fluid and $\beta>0$ represents the Jeffrey fluid parameter. However, opposing phenomenon is observed for the temperature profile. The effect of $\gamma$ is found to decrease the velocity distribution and increase temperature distribution, respectively.

The effects of the MHD parameter $M$ on the velocity $f'(\eta)$ and the temperature profiles $\theta(\eta)$ are shown in Figures 7 and 8, respectively. Velocity is found to decrease with the increase of $M$. The introduction of the transverse magnetic field will result in a restrictive force (Lorenz force), which tends to resist the motion of the fluid flow and hence, lead to the decrement of velocity. However, the opposite trend is observed in the increment of $M$, which results in the increment of temperature distribution across the boundary layer. The effect of the porous medium $\gamma$ on flow velocity and temperature can also be garnered from the same figures. It is obvious that an increase in the porosity $\gamma$ causes greater obstruction to the fluid flow, which culminates in the decrement of velocity, whereas the opposite
trend occurs for the temperature profile $\theta(\eta)$, i.e the increment of $\gamma$ results in an increment in temperature and thermal boundary layer thickness, as shown in Figures 7 and 8, respectively.

Figures 9 and 10 present the velocity and temperature profiles when $Pr=0.7$ and 6.8 for few values of the mixed convection parameter $\lambda$, respectively. It is well known that $\lambda=0$ corresponds to pure forced convection and the presence of thermal buoyancy ($\lambda \neq 0$) will lead to stronger buoyancy force, which induces more flow along the surface. The consequences can be seen in the increase of the velocity $f'(\eta)$ as $\lambda$ increases. However, this phenomenon is more pronounced for flow with low $Pr$ numbers compared to flow with high $Pr$ numbers. An overshoot peak in the velocity profile is observed near the surface for flow with low $Pr$ number and for large values of the mixed convection parameter ($\lambda=10$) where the free convection is dominant. At the beginning of the motion ($0 \leq \eta \leq 0.5$), the velocity increases until it reaches a certain value and gradually decreases until it goes to 0 at the outside of the boundary layer, whereas the velocity for other profiles produce lower velocities toward the edge of the boundary layer starting from the beginning.

Figure 10 depicts the graph of the temperature distributions for the same data used in Figure 9. The tabulated temperature is more noticeable for different values of $\lambda$ when $Pr=0.7$ compared to $Pr=6.8$. The aim of the increasing the values of $\lambda$ is to decrease the thickness of the thermal boundary layer and reduce temperature. However, this phenomenon does not happen for $Pr=6.8$, i.e the variations of $\lambda$ appear not to influence temperature distribution as they are seen to have similar profiles.
Irrespective of the value of the parameters in this study, all the plotted velocity and temperature profiles satisfied the boundary conditions (8) asymptotically.

4. Conclusions

The present study considered the steady MHD flow and heat transfer of Jeffrey fluid over a stretching sheet towards vertical sheet embedded in porous media. The effects of the Deborah number $\beta$, magnetic parameter $M$, porosity parameter $\gamma$ and Prandtl number $Pr$ are numerically studied and some graphs for the skin friction coefficient and the local Nusselt number, along with velocity and temperature profiles, are plotted for these reasons. The magnetic parameter $M$ has an important effect to the heat transfer processing; i.e. increment of $M$ decreases the heat transfer rate. While, the heat transfer increases as $Pr$ increases. Flow of Jeffrey fluid is found to decrease the magnitude of the skin friction and slightly increases the heat transfer rate at the surface.

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References
