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Analytical approximate solutions of the duffing-relativistic oscillator

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***Abstract*—The aim of this paper is to use high-order harmonic balance method (HBM) as a novel solution procedure for investigation of the Duffing-relativistic oscillation. Usually, a set of complex nonlinear algebraic equations is appeared when HBM is applied. Investigating analytically for such kind of complex nonlinear algebraic equations is tremendously difficult and cumbersome. In the present study, a small parameter is found, for which the power series solutions produces desired results. The obtained results are evaluated by comparing with the exact solutions computed numerically. The effect of initial conditions in the nonlinear natural frequencies is carried out and it is proved the proposed method is not only simple, but also more reliable for analysis for the Duffing-relativistic oscillator. The method is mainly illustrated in strongly nonlinear Duffing-relativistic oscillator, but it can be widely applicable in other problems arising nonlinear sciences and engineering.**

***Keywords*—duffing-relativistic oscillator, harmonic balance method, power series solutions**

I. Introduction

The nonlinear Duffing-relativistic oscillator has received considerable attention, especially in the last decade in nonlinear sciences and engineering. Along with the rapid progress of nonlinear sciences, an intensifying interest among scientists and researchers invest their effort to developed varieties of analytical approximate and numerical solution methods to solve Duffing-relativistic oscillator. A large variety of variational and perturbative methods commonly used for nonlinear oscillatory systems, especially for strongly nonlinear oscillators have been recently extended for instance, one can refer to the Modified He's Homotopy Perturbation Method [1-3], He's Modified Lindsted-Poincare Method [4], He's Max-Min Approach Method [5], He's Energy Balance Method [6-9], He's Frequency Amplitude Formulation [10-11] and other classical perturbative and non-perturbative techniques including Homotopy Perturbation Method [12], Elliptic Balance Method [13], Algebraic Method [14], Rational Energy Balance Method [15], Iteration Method [16-18], Residue Harmonic Balance Method [19], Hamiltonian Approach Method [20], Rational Harmonic Balance Method [21] and so on. Harmonic balance method (HBM) is another method for solving strongly nonlinear oscillator [22-29]. However, most of these methods only considered first-order approximation solution which leads to a low accuracy. In addition, the aforementioned methods also do not have ability to gain the solution in high precision. Furthermore, the solution procedures are tremendously difficult task and cumbersome, especially, for obtaining a higher order approximation. In this situation, approximate periodic solutions for the Duffing-relativistic oscillator are studied by employing HBM. The second-order approximate solution has been obtained for the Duffing-relativistic oscillator. The proposed technique not only provides accurate results, but also it is more convenient and effective for solving more complex nonlinear problems. Error analysis is then carried out and performance of the solution technique is compared with exact ones.

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II. Problem Descriptions

The equation of Duffing-relativistic oscillator is of the form

$$\ddot{x} + x + x^3 - \frac{\beta x}{\sqrt{x^2+1}} = 0\,, \qquad (1)$$

where, over dot denotes differentiation with respect to time t and $0 < \beta \le 1$. For $\beta = 0$ the equation governs as a type of the well know Duffing oscillator which represents free undamped vibration of an orthotropic clamped triangular plate [8].

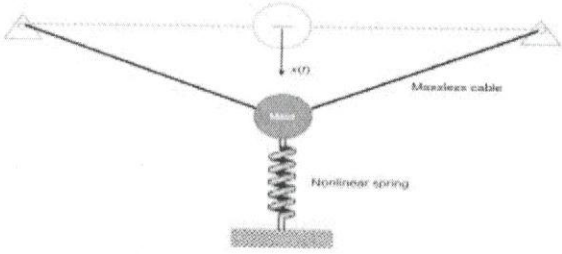

Figure 1: Schematic of a Duffing-relativistic oscillator.

III. The method

Let us consider a second-order nonlinear differential equation

$$\ddot{x} + \omega_0^2 x = -\varepsilon f(x), \quad [x(0) = A_0, \dot{x}(0) = 0] \qquad (2)$$

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where, $f(x)$ is a nonlinear function such that, $f(-x)=-f(x)$, $\omega_0 \geq 0$ and ε is a constant. Considering a periodic solution of "(2)" is in the form

$$x = A_0(\rho\cos(\omega t)+u\cos(3\omega t)+v\cos(5\omega t) + w\cos(7\omega t)+z\cos(9\omega t)\cdots) \tag{3}$$

where, A_0 , ρ and ω are constants. If $\rho = 1-u-v-\cdots$ and the initial phase value $(\omega t)_0 = 0$, solution "(3)" readily satisfies the initial conditions $[x(0)=A_0,\, \dot{x}(0)=0]$.

Substituting "(3)" into "(2)" and expanding $f(x)$ in a Fourier series, it converts to an algebraic identity as

$$\begin{aligned} &A_0[\rho(\omega_0^2-\omega^2)\cos(\omega t)+u(\omega_0^2-9\omega^2)\cos(3\omega t)+\cdots] \\ &= -\varepsilon[F_1(A_0,u,\cdots)\cos(\omega t)+F_3(A_0,u,\cdots)\cos(3\omega t)+\cdots] \end{aligned} \tag{4}$$

By comparing the coefficients of equal harmonics of "(4)", the following nonlinear algebraic equations are found

$$\begin{aligned} &\rho(\omega_0^2-\omega^2)=-\varepsilon F_1, \quad u(\omega_0^2-9\omega^2)=-\varepsilon F_3, \\ &v(\omega_0^2-25\omega^2)=-\varepsilon F_5, \cdots \end{aligned} \tag{5}$$

With the help of the first equation, ω^2 is eliminated from all the rest of "(5)". Thus "(5)" takes the following form

$$\begin{aligned} &\rho\omega^2=\rho\omega_0^2+\varepsilon F_1, \quad 8\omega_0^2 u\rho=\varepsilon(\rho F_3-9uF_1), \\ &24\omega_0^2 v\rho=\varepsilon(\rho F_5-25vF_1), \cdots \end{aligned} \tag{6}$$

Substitution $\rho = 1-u-v-\cdots$, and simplification, second-, third- equations of "(6)" take the following form

$$\begin{aligned} &u=G_1(\omega_0,\varepsilon,A_0,u,v,\cdots,\lambda_0), \\ &v=G_2(\omega_0,\varepsilon,A_0,u,v,\cdots,\lambda_0), \cdots \end{aligned} \tag{7}$$

where, $G_1, G_2, \cdots$ exclude respectively the linear terms of $u, v, \cdots$.

Whatever the values of ω_0 , ε and A_0 , there exists a parameter $\lambda_0(\omega_0,\varepsilon,A_0) \ll 1$, such that $u, v, \cdots$ are expandable in following power series in terms of λ_0 as

$$u=U_1\lambda_0+U_2\lambda_0^2+\cdots, \quad v=V_1\lambda_0+V_2\lambda_0^2+\cdots, \quad \cdots \tag{8}$$

where, $U_1, U_2, \cdots, V_1, V_2, \cdots$ are the constants.

Finally, substituting the values of $u, v, \cdots$ from "(8)" into the first equation of "(6)" for determining the natural frequency ω . This completes the determination of all related functions for the proposed periodic solution as given in "(3)".

IV. Example

A. *Duffing-relativistic oscillator*

For a Duffing-relativistic oscillator it be can written as

$$\ddot{x}+x+x^3-\frac{\beta x}{\sqrt{x^2+1}}=0, \tag{9}$$

with the initial condition is, $[x(0)=A_0,\, \dot{x}(0)=0]$
Here, substitution of approximation

$$\frac{\beta x}{\sqrt{x^2+1}}=\beta\left(x-\frac{x^3}{2}+\frac{3x^5}{8}-\frac{5x^7}{16}+\cdots\right) \tag{10}$$

into "(9)" yields,

$$\ddot{x}+x+x^3-\beta\left(x-\frac{x^3}{2}+\frac{3x^5}{8}-\frac{5x^7}{16}+\cdots\right)=0 \tag{11}$$

From "(3)" the first-order approximation solution of "(11)" is given by "(12)".

$$x=A_0\cos(\omega t) \tag{12}$$

Now substituting "(12)" into "(11)", taking $\beta=1$ and setting the coefficient of $\cos(\omega t)$ the following algebraic equation is obtained

$$1152A_0^2-240A_0^4+175A_0^6-1024\omega^2=0 \tag{13}$$

Thus, from "(13)" the first-order approximate natural frequency can be written as "(14)".

$$\omega=\sqrt{\frac{9A_0^2}{8}-\frac{15A_0^4}{64}+\frac{175A_0^6}{1024}} \tag{14}$$

Therefore, the first-order approximation solution of "(9)" is "(12)" *i.e.*, $x=A_0\cos(\omega t)$ where ω is given by "(14)".

Now considering a second-order approximation solution

$$x=A_0\cos(\omega t)+A_0u(\cos(3\omega t)-\cos(\omega t)) \tag{15}$$

Substituting "(15)" into "(11)" taking $\beta=1$ and then equating the coefficients of $\cos(\omega t)$ and $\cos(3\omega t)$, the following nonlinear algebraic equations are obtained

$$\begin{aligned} &\frac{9A_0^2}{8}-\frac{15A_0^4}{64}+\frac{175A_0^6}{1024}-\frac{9A_0^2u}{4}+\frac{75A_0^4u}{128}-\frac{245A_0^6u}{512} \\ &+\frac{27A_0^2u^2}{8}-\frac{45A_0^4u^2}{32}+\frac{735A_0^6u^2}{512}-\frac{9A_0^2u^3}{4}+\frac{75A_0^4u^3}{32} \\ &-\frac{875A_0^6u^3}{256}-\frac{75A_0^4u^4}{32}+\frac{6125A_0^6u^4}{1024}+\frac{135A_0^4u^5}{128} \\ &-\frac{3675A_0^6u^5}{512}+\frac{2695A_0^6u^6}{512}-\frac{455A_0^6u^7}{256}-\omega^2+u\omega^2=0, \end{aligned} \tag{16}$$

Similarly,

$$\begin{aligned} &\frac{3A_0^2}{8}-\frac{15A_0^4}{128}+\frac{105A_0^6}{1024}+\frac{9A_0^2u}{8}-\frac{15A_0^4u}{128}-\frac{27A_0^2u^2}{8} \\ &+\frac{15A_0^4u^2}{16}-\frac{315A_0^6u^2}{512}+3A_0^2u^3-\frac{75A_0^4u^3}{32}+\frac{2625A_0^6u^3}{1024} \\ &+\frac{375A_0^4u^4}{128}-\frac{6125A_0^6u^4}{1024}-\frac{195A_0^4u^5}{128}+\frac{2205A_0^6u^5}{256} \\ &-\frac{3675A_0^6u^6}{512}+\frac{1365A_0^6u^7}{5128}-9u\omega^2=0 \end{aligned} \tag{17}$$

After simplification, "(16)" takes the form

$$\omega^2=\left(\frac{9A_0^2}{8}-\frac{15A_0^4}{64}+\frac{175A_0^6}{1024}-\frac{9A_0^2u}{4}+\frac{75A_0^4u}{128}+\cdots\right)/(1-u) \tag{18}$$

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By elimination of ω^2 from "(17)" with the help of "(18)" and simplification, the following nonlinear algebraic equation for u can be found from "(19)" as follows:

$$u=\lambda_0\left(-\frac{3}{8}+\frac{15A_0^2}{128}-\frac{105A_0^4}{1024}-\frac{63u^2}{4}+\frac{135A_0^2u^2}{32}+24u^3+\cdots\right), \quad (19)$$

where, $\lambda_0=\dfrac{1}{\left(-\frac{75}{8}+\frac{135A_0^2}{64}-\frac{105A_0^4}{64}\right)}$.

The solution of "(20)" in terms of λ_0 is

$$u=\left(-\frac{3}{8}+\frac{15A_0^2}{128}-\frac{105A_0^4}{1024}\right)\lambda_0 +\left(-\frac{567}{256}+\frac{2025A_0^2}{1024}+\cdots\right)\lambda_0^3 +\left(-\frac{81}{64}+\frac{6885A_0^2}{4096}+\cdots\right)\lambda_0^4+\cdots \quad (20)$$

Now, substituting the value of u from "(20)" into "(18)", the second-order approximate natural frequency ω is determined. Thus the second-order approximation solution of "(9)" is $x=a_0\cos(\omega t)+a_0u(\cos(3\omega t)-\cos(\omega t))$ where u and ω are respectively given by "(20)" and "(18)".

II. Results and discussions

The first and second-order approximate solutions and their relative errors (RE) obtained from this article by applying HBM for the Duffing-relativistic oscillator is shown in Table 1, Table 2, Figure 1 and Figure 2. It illustrates the accuracy of the HBM method by comparing with the numerical forth order Runge-Kutta results and also, analytical error between these quantities for two numerical cases in each 0.5 sec. It can clearly be seen that the accuracy of the solutions (second-order approximation) obtained by the proposed technique is very close to the exact solutions. It is noted that the solution procedure of the proposed method is simple, straightforward, quite easy and highly efficient.

Table 2: First- and second-order approximate solutions of "(1)" and compared with numerical forth order Runge-Kutta method for $A_0=\pi/18$

t	$x^{(1)}$	$x^{(2)}$	x_{nu}	% RE of $x^{(1)}$	% RE of $x^{(2)}$
0.0	0.1745	0.1745	0.1745	0.0000	0.0000
0.5	0.1738	0.1736	0.1735	0.0003	0.0001
1.0	0.1716	0.1707	0.1706	0.0010	0.0001
1.5	0.1679	0.1661	0.1658	0.0021	0.0003
2.0	0.1628	0.1597	0.1594	0.0034	0.0003
2.5	0.1563	0.1519	0.1514	0.0049	0.0005
3.0	0.1485	0.1426	0.1421	0.0064	0.0005
3.5	0.1394	0.1323	0.1318	0.0076	0.0005
4.0	0.1291	0.1210	0.1206	0.0085	0.0004
4.5	0.1177	0.1090	0.1087	0.0090	0.0003
5.0	0.1054	0.0965	0.0964	0.0090	0.0001

Table 2: First- and second-order approximate solutions of "(1)" and compared with numerical forth order Runge-Kutta method for $A_0=\pi/6$

t	$x^{(1)}$	$x^{(2)}$	x_{nu}	RE of $x^{(1)}$	RE of $x^{(2)}$
0.0	0.5236	0.5236	0.5236	0.0000	0.0000
0.5	0.5045	0.4993	0.4988	0.0057	0.0005
1.0	0.4484	0.4316	0.4306	0.0178	0.0010
1.5	0.3596	0.3339	0.3336	0.0260	0.0003
2.0	0.2445	0.2213	0.2225	0.0220	0.0012
2.5	0.1114	0.1049	0.1069	0.0045	0.0020
3.0	-0.0297	-0.0112	-0.0094	0.0203	0.0018
3.5	-0.1687	-0.1274	-0.1258	0.0429	0.0016
4.0	-0.2953	-0.2436	-0.2410	0.0543	0.0026
4.5	-0.4004	-0.3544	-0.3506	0.0498	0.0038
5.0	-0.4762	-0.4475	-0.4440	0.0322	0.0035

Note: $x(^1)$ and $x^{(2)}$ respectively denote the first and second-order approximate solutions obtained by *HBM*. Similarly, x_{nu} represents the numerical forth order Runge-Kutta solution and relative error is denoted by *RE* of $x=|(x_{nu}-x(^i))|$ where $i=1,2$.

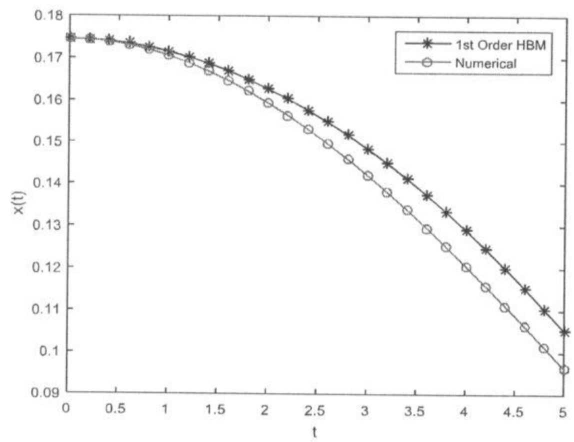

Figure 2. Graphical comparison of first-order approximate solutions of "(1)" and with numerical forth order Runge-Kutta method for $A_0=\pi/18$

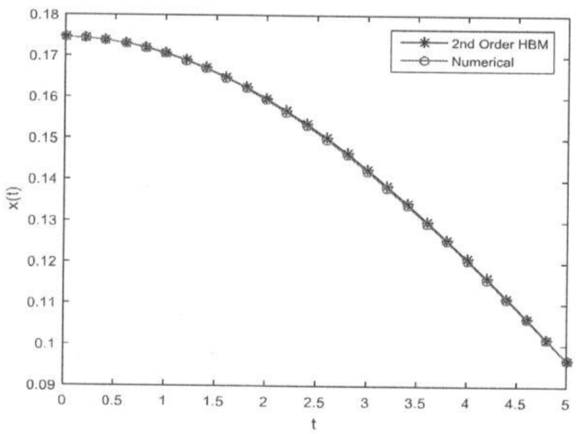

Figure 3. Graphical comparison of second-order approximate solutions of "(1)" and with numerical forth order Runge-Kutta method for $A_0=\pi/18$

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The advantages of this method include its analytical simplicity and computational efficiency, and the ability to objectively find better results for many other oscillatory problems arising in nonlinear sciences and engineering. Figure 2 and Figure 3 represent the graphical comparison between approximate solutions of "(1)" and with numerical forth order Runge-Kutta method ($A_0 = \pi/18$) for first-order and second-order respectively. From these figures, it is found that the second-order approximate solutions of "(1)" is more closer to the numerical forth order Runge-Kutta method.

V. Conclusion

Approximate periodic solutions for the Duffing-relativistic oscillator were analytically obtained using HBM. Periodic solutions and natural frequencies were analytically studied. Error analysis was also carried out and it was found that the proposed method lead to more accurate solutions. The high accuracy and validity of approximate solutions assured us about the solution and reveal the method can be used for strongly nonlinear oscillators even with a higher order of nonlinearity. To sum up, we can say that the method introduced in this study for solving Duffing-relativistic oscillator can be considered as powerful, an efficient alternative of the previously existing methods.

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