A FEW REMARKS ON RELATIVE ERGODIC PROPERTIES OF C*-DYNAMICAL SYSTEMS

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Abstract. We study various ergodic properties of C*-dynamical systems inspired by unique ergodicity. In particular we work in a framework allowing for ergodic properties defined relative to various subspaces, and in terms of weighted means.

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1. Introduction

The study of ergodic theorems in recent years showed that the ordinary Cesaro means have been replaced by weighted averages

$$\sum_{k=0}^{n-1} a_k f(T^k x). \tag{1}$$

Therefore, it is natural to ask: is there a weaker summation than Cesaro, ensuring the unique ergodicity. In [42] it has been established that unique ergodicity implies uniform convergence of (1), when $\{a_k\}$ is Riesz weight (see also [38] for similar results). In this paper we are going to study such kind of problem in general setting. Moreover, we also investigate related notions such as mixing etc.

It is known [14] that the theory of quantum dynamical systems provides a convenient mathematical description of irreversible dynamics of an open quantum system investigation of ergodic properties of such dynamical systems have had a considerable growth. In a quantum setting, the matter is more complicated than in the classical case. This motivates an interest to study dynamics of quantum systems (see [27, 28, 29]). Therefore, it is then natural to address the study of the possible generalizations to quantum case of various ergodic properties known for classical dynamical systems. In [8, 46, 51] a non-commutative notion of unique ergodicity was defined, and certain properties were studied. Recently in [2] a general notion of unique ergodicity for automorphisms of a C^*-algebra with respect to its fixed point subalgebra has been introduced. In [3] a generalization of such a notion for positive mappings of C^*-algebras, and its characterization in term of Riesz means are given. When studying ergodic properties of such a system, it has become clear that it is often necessary to work relative to some subalgebra (or even some more general subspace) of the C*- or W*-algebra involved (see for example [52],[13],[7]).

In this paper, we study various ergodic properties of C*-dynamical systems for semigroup actions and in terms of weighted means. The properties are to a large extent inspired by the notion of unique ergodicity relative to the fixed point space as introduced in [2], but of a more general form, for example allowing one to work relative to other spaces than just the fixed point space.

2. Weighted means

Definition 2.1. Let G be a topological semigroup with a right invariant measure ρ on its Borel σ-algebra, and let X be a Banach space. Consider a net $(f_\iota) \equiv (f_\iota)_{\iota \in I}$ indexed by some directed set I, where $f_\iota \in L^1(\rho)$, $f_\iota : G \to \mathbb{R}^+ = [0, \infty)$ and $\int f_\iota d\rho \neq 0$. Assume furthermore that $f_\iota F$ is Bochner integrable for all bounded Borel measurable $F : G \to X$, and that for such F

$$\lim_\iota \frac{\int f_\iota(g)\left[F(g) - F(gh)\right] dg}{\int f_\iota d\rho} = 0$$

in the norm topology for all $h \in G$ (where dg refers to integration with respect to the measure ρ). Then we call (f_ι) a *(right) averaging net* for (G, X). If we rather require the condition

$$\lim_\iota \frac{\int f_\iota(g)\left[F(g) - F(hg)\right] dg}{\int f_\iota d\rho} = 0$$

for all $h \in G$, then we call (f_ι) a *left averaging net* for (G, X)

Proposition 2.2. *Let G be a topological semigroup with a right invariant measure ρ on its Borel σ-algebra. Let (Λ_ι) be a Følner net in G, i.e. Λ_ι is a compact set in the Borel σ-algebra of G with $0 < \rho(\Lambda_\iota) < \infty$ and*

$$\lim_\iota \frac{\rho(\Lambda_\iota \triangle (\Lambda_\iota g))}{\rho(\Lambda_\iota)} = 0$$

for all $g \in G$. Let $f_\iota := \chi_{\Lambda_\iota}$ be the characteristic (i.e. indicator) function of Λ_ι on G. Then (f_ι) is an averaging net for (G, X) for any Banach space X.

Proposition 2.3. *Let G be a topological group with a right invariant measure ρ on its Borel σ-algebra, and let X be a Banach space. Consider a net $(\Lambda_\iota, f_\iota) \equiv (\Lambda_\iota, f_\iota)_{\iota \in I}$ indexed by some directed set I, where $\Lambda_\iota \subset G$ is Borel measurable, and $f_\iota \in L^1(\Lambda_\iota)$ (in terms of ρ restricted to Λ_ι), such that $f_\iota : \Lambda_\iota \to \mathbb{R}^+$ and $\int_{\Lambda_\iota} f d\rho \neq 0$. Assume furthermore that*

$$\lim_\iota \frac{\int_{\Lambda_\iota \setminus (\Lambda_\iota h)} f_\iota d\rho}{\int_{\Lambda_\iota} f_\iota d\rho} = 0 \text{ and } \lim_\iota \frac{\int_{\Lambda_\iota \cap (\Lambda_\iota h)} \left| f_\iota(g) - f(gh^{-1}) \right| dg}{\int_{\Lambda_\iota} f_\iota d\rho} = 0$$

for all $h \in G$. Define a function f'_ι on G by $f'_\iota(x) = f_\iota(x)$ for $x \in \Lambda_\iota$, and $f'_\iota(x) = 0$ for $x \notin \Lambda_\iota$. Then (f'_ι) is an averaging net for (G, X) for any Banach space X.

The examples below (for $G = \mathbb{R}$) can be checked by using the proposition above.

Example 2.4. Consider the case $G = \mathbb{R}$. Set $\Lambda_n := [0, n]$ for $n = 1, 2, 3, ...$, or even any real $n > 0$. Let $f(t) := t^s$ for an $s > -1$. Setting $f_n := f|_{\Lambda_n}$, one can verify that (Λ_n, f_n) gives an averaging net for $\mathbb{R}$ as in Proposition 2.3.

Example 2.5. Similarly $\Lambda_n := [1, n]$ for $n = 2, 3, ...$, or even any real $n > 1$, along with $f(t) = t^{-1}$, gives an averaging net for $\mathbb{R}$.

Example 2.6. Lastly, $\Lambda_n := [0, n]$ and $f_n(t) := (n - t)^s$ for $s > -1$, give an averaging net for $\mathbb{R}$.

Theorem 2.7. *Consider a topological semigroup G with a right invariant measure ρ on its Borel σ-algebra, a Hilbert space H, and an averaging net (f_ι) for (G, H). Let U be a representation of G as contractions on H, such that $G \to H : g \mapsto U_g x$ is Borel measurable for all $x \in H$. Let P be the projection of H onto the fixed point space V of U, namely*

$$V := \{x \in H : U_g x = x \text{ for all } g \in G\} .$$

Then

$$\lim_\iota \frac{1}{\int f_\iota d\rho} \int_{\Lambda_\iota} f_\iota(g) U_g x dg = Px$$

for all $x \in H$.

Definition 2.8. A *C*-dynamical system* (A, α) consists of a unital C*-algebra A and an action α of a semigroup G on A as unital completely positive maps $\alpha_g : A \to A$, i.e. as Markov operators. The *fixed point operator system* of a C*-dynamical system (A, α) is defined as

$$A^\alpha := \{a \in A : \alpha_g(a) = a \text{ for all } g \in G\}.$$

By an *operator system* of A, we mean a norm closed self-adjoint vector subspace of A containing the unit of A. Whenever we consider a C*-dynamical system (A, α), the notation G for the semigroup is implied. Note that since α_g is positive and $\alpha_g(1) = 1$, we have $\|\alpha_g\| = 1$.

Definition 2.9. A C*-dynamical system (A, α) is called *amenable* if the following conditions are met: G is a topological semigroup with a right invariant measure ρ on its Borel σ-algebra, and furthermore (G, A) has an averaging net (f_ι). The function $G \to A : g \mapsto \alpha_g(a)$ is Borel measurable for every $a \in A$.

A central notion in our work will be that of an invariant state:

Definition 2.10. Given a C*-dynamical system (A, α), a state μ on A is called an *invariant* state of (A, α), or alternatively an *α-invariant* state, if $\mu \circ \alpha_g = \mu$ for all $g \in G$.

Definition 2.11. We call the C*-dynamical system (A, α) *uniquely ergodic relative to* A^α if every state on A^α has a unique extension to an invariant state of (A, α).

Theorem 2.12. *Let (A,α) be an amenable C^*-dynamical system, with G unimodular with respect to the measure ρ, and let (f_ι) be both a right and left averaging net for (G,A). Then statements (i) to (vi) below are equivalent.*

(i) *The system (A,α) is uniquely ergodic relative to A^α.*

(ii) *The limit*

$$\lim_\iota \frac{1}{\int f_\iota d\rho} \int f_\iota(g)\, \alpha_g(a) dg$$

exists for every $a \in A$.

(iii) *The subspace $A^\alpha +$ span$\{a - \alpha_g(a) : g \in G, a \in A\}$ is dense in A.*

(iv) *The equality $A = A^\alpha + \overline{\text{span}\{a - \alpha_g(a) : g \in G, a \in A\}}$ holds.*

(v) *Every bounded linear functional on A^α has a unique bounded α-invariant extension to A with the same norm.*

(vi) *There is a positive projection E of A onto some operator system B of A such that for every $a \in A$ and $\varphi \in S(A)$, where $S(A)$ denotes the set of all states on A, one has*

$$\lim_\iota \frac{1}{\int f_\iota d\rho} \int f_\iota(g)\, \varphi(\alpha_g(a)) dg = \varphi(E(a))$$

(in which case necessarily $B = A^\alpha$ and $\alpha_g \circ E = E = E \circ \alpha_g$ for all $g \in G$).

Furthermore, statements (i) to (vi) imply the following statements:

(vii) *There exists a unique α-invariant positive projection E from A onto A^α.*

(viii) *The positive projection E in (vii) is given by*

$$Ea = \lim_\iota \frac{1}{\int f_\iota d\rho} \int f_\iota(g)\, \alpha_g(a) dg$$

for all $a \in A$.

Let (A, α) be a C^*-dynamical system, let $E : A \to A$ be a bounded linear operator, and let S be a set of bounded linear functionals on A. In what follows, by $S(A)$ we denote the set of all states defined on A.

Definition 2.13. A C^*-dynamical system (A, α) for the action of a topological semigroup G is called *S-weakly amenable*, for a set $S \subset A^*$, if the following holds: There is a right invariant measure ρ on G, an averaging net (f_ι) for $(G, \mathbb{C})$, and $G \to \mathbb{C} : g \mapsto \varphi(\alpha_g(a))$ is Borel measurable for every $a \in A$ and $\varphi \in S$.

Definition 2.14. Let (A, α) be an S-weakly amenable C^*-dynamical system, with G unimodular with respect to the right measure ρ and let (f_ι) be an averaging net for $(G, \mathbb{C})$. Then (A, α) is said to be

(i) *unique (E, S)-ergodic* w.r.t. (f_ι) if one has

$$\lim_\iota \frac{1}{\int f_\iota} \int f_\iota(g)\varphi(\alpha_g(x - E(x)))dg = 0\,, \quad x \in A\,, \varphi \in S\,; \tag{2}$$

(ii) *unique (E, S)-weakly mixing* w.r.t. (f_ι) if one has

$$\lim_\iota \frac{1}{\int f_\iota} \int f_\iota(g)\big|\varphi(\alpha_g(x - E(x)))\big|dg = 0\,, \quad x \in A\,, \varphi \in S\,. \tag{3}$$

Theorem 2.15. *Let (A, α) be an $\mathcal{S}$-weakly amenable C^*-dynamical system and let (f_ι) be an averaging net for $(G, \mathbb{C})$. Let the dynamical system $(A \otimes A, \alpha \otimes \alpha)$ be unique $(E \otimes E, \mathcal{S} \otimes \mathcal{S})$-ergodic, and assume that E preserves the involution (i.e. $E(x^*) = E(x)^*$) and that $\mathcal{S}$ is self-adjoint. Then (A, α) is unique $(E, \mathcal{S})$-weakly mixing.*

Remark 2.16. We note that in [8, 47, 56] similar results were proved for weak mixing dynamical systems defined over von Neumann algebras.

Theorem 2.17. *Let (A, α) and (B, β) respectively be an $\mathcal{S}$-weakly amenable and an $\mathcal{H}$-weakly amenable C^*-dynamical system, and let (f_ι) be an averaging net for $(G, \mathbb{C})$. If (A, α) is unique $(E_\alpha, \mathcal{S})$-weakly mixing and (B, β) unique $(E_\beta, \mathcal{H})$-ergodic with $\alpha_g E_\alpha = E_\alpha$ for all $g \in G$, then the C^*-dynamical system $(A \otimes B, \alpha \otimes \beta)$ is unique $(E_\alpha \otimes E_\beta, \mathcal{S} \otimes \mathcal{H})$-ergodic.*

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