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Using the Jar Model to Improve Students' Understanding of Operations on Integers

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The focus of this paper is to report on a study that assess students' knowledge and understanding of integers before and after the intervention teaching using the 'jar model'. The paper will concentrate on the kind of errors students make in learning integers and how the 'jar model' was supposed to enhance students' understanding instead of memorising rules like 'negative times negative gives positive' etc. Analyses from interviews and performance data of the pre and post-intervention stage revealed that most of the students can understand the jar model and thus improvement can be seen from the result of the post-test.

Keywords: integers, integers operations, jar model, positive and negative numbers

1 Introduction

Students' conceptions of the nature of mathematics and their approaches to studying mathematics have long been matters of interest to mathematics education researchers, largely because both are believed to have an impact on the quality of students' mathematics learning. As is the case in many other parts of the world, educators in Brunei Darussalam are concerned that too many secondary school students pass mathematics examinations without really understanding the subject. Often, students appear to believe that mathematics is a mechanical, rule-bound discipline (Noridah, 1999).

Studies done by Zurina (2003), Khoo (2001), Lim (2000) and Noridah (1999) indicated that the secondary school students in Brunei Darussalam had been taught by methods which emphasize drill and practice with the focus on preparation for the test or examination. Most teachers felt the pressure to move through the mathematics syllabuses as quickly as possible, in order to have the extra time to prepare students for tests and examinations. From the teachers' perspective, this meant that little time is available to attempt to teach for conceptual understanding. For them, teaching for understanding was fine and ideal, but examinations are more crucial. The reason for this is because teachers are usually judged by the examination results and good teachers are usually those who produced good results. On this view, many teachers teach the students for the sake of passing examinations instead of emphasizing understanding of concepts.

Based on one of the researcher's experience of teaching integers in lower secondary government school, it was found that many students were facing difficulties in understanding the topic of integers. Through informal observations and conversations, many secondary mathematics teachers in Brunei Darussalam have expressed their concern over students' poor performances on integers. The teachers indicated that although they recognise that many of their students do not like integers and struggle with integers' questions, they do not know what to do to improve the situation. They tend to say that their students get confused with the signs and operations on integers although they had attempted to explain about it several times. The current number line model that is popular among the teachers is also confusing to the students. The main problems in teaching for understanding of negative numbers and operations are in developing effective strategies for adding, subtracting, multiplying and dividing integers. Some students do not even know how to determine whether one integer is greater than, less than or equal to another integer.

This paper will further describe other studies done to enhance better understanding of integers in other countries, the methodology employed in the research, the findings from the pre-test (the kinds of errors common among the students) and the result of teaching using the jar model on students' understanding of the subject.

2 Literature review

2.1 Research on Teaching and Learning of Integers

Many articles and research papers were found to describe studies in which teachers and students used different strategies in the teaching and learning of integers. Developing effective teaching strategies of integers has been ongoing in many parts of the world. In order to make students understand integers we have to extend their knowledge, help them make logical connections with what they know and use appropriate strategies in learning. Papers have been written on the teaching and learning of integers by Jenny (2002), Dehaene (1997), Hayes (1999), Hart et al (1981), Freudenthal (1973) etc..

Hayes (1996) conducted research on the effectiveness of the most common strategies for negative number concepts and operations at three secondary schools involving students in years seven, eight and nine. The experimental teaching groups used reversible two centimeter square tiles labeled [+1], [–1] and [0]. The major difference in strategy between the experimental and control groups was that the experimental groups started with the tiles. By the end of the topic the experimental group students had also used the number line in context of ordering and 2D point plotting. The outcomes of the study, in terms of student short and long term performance have been compared with those in classes taught by more commonly used strategies. The experimental approach seems to have facilitated better performances for average ability level students. For more able mathematics students, the topic does not appear to be difficult and such students, in both experimental and control groups indicated good levels of general topic mastery. Hayes (1996) found that the use of these tiles led to a significant improvement on a range of test items, including examples requiring the use of brackets, mixed operations and order of operations. It also developed a more confident and secure knowledge of the rules and showed fewer tendencies to confuse sign rules across operations. This study coincides with that of Linchenvski and Williams (1988) involves teaching negative numbers using teaching aids, that is, by using dice with +3, +2, +1, –3, –2 and –1 painted on them. They found that the students soon started to use cancellation and compensation strategies.

Chinese Yin/Yang is one of the examples used by Egan (1997) for teaching and learning of directed number which is quite common in Chinese society. A similar approach has also been observed in a Taiwan book for mathematics educators. Teachers are usually advised to ask students to produce several +1 and –1 figures, so that they can play with the figures to explore the principle of addition and subtraction, then multiplication and division (for the purpose of demonstrating division, some teachers may prefer the use of +4 and –4 instead). This design of teaching and learning activities for directed number is not only effective, but it has its implication on the use of metaphor and students' development of Mythic Understanding (Tang, 2003). The Chinese "Tai ji" (or Yin-yan) symbol consists of two parts: light (yin) and shadow (yan). The light part represents warm and bright sides of the nature while the shadow part represents the cold and dark sides. Thus, the light part can be treated as 'positive' and the shadow part 'negative'. These two parts, when grouped together, have a meaning of balance and harmony. If we use +1 to replace light and –1 to replace shadow (see Fig. ii), the whole diagram now represents the number zero. Teachers can produce several +1 and –1 figures using thick stiff cardboard and use them to explain the principle of addition and subtraction of directed numbers.

Angela (2003) did a study on teaching negative numbers using multi-link cubes with a Year 7 mixed ability group. In her study she found that the students had a physical representation of a negative number but this did not help them to understand what a negative number is. It did not help them when it came to understanding why subtracting a negative number would make the answer bigger. However, the cubes could help students calculate a correct answer but they had no reasoning to help them know that the answer was correct. The multi-link cubes only help students as a counting aid, and once the aid was removed they struggled to answer the questions set. She then furthered her study on a Year 8 group with the top ability group. Though these students had been taught the 'rules of negative numbers' the previous year, they still made common errors by exchanging two negative signs for one negative sign rather than a positive sign. In her study, she chose the context of hot and cold water as she felt that the context of temperature would be something 'real' to all students. Though some students were still confused by the context, it seemed that they were all confident with the temperature idea. Most of the students could answer the questions correctly. This indicated that they had a much more thorough understanding through the use of the temperature context.

It is important for teachers to assess the appropriate ways to teach negative numbers and evaluate the student's understandings. Apart from using mathematical resources to teach, it is also important to teach mathematics with a focus on number sense. These will encourage students to become problem solvers in a wide variety of situations and view mathematics as a discipline in which thinking is important.

2.2 The 'Jar Model'

The method resembling the jar model has been used by Battista (1983) and Paul Griffifth (2002) for teaching integers. In fact, similar models were advocated and can be found at websites such as Homeschool Math (2003) and Learning Math (2002). Basically, this method is not very different from other methods that had been used by some of the researchers mentioned in the literature review (Jenny, 2002; Tang, 2003; Hayes, 1996; Egan, 1997). However, effort was taken to consider students' cultural situation and social context. Please refer to Appendix 1 for the some explanation about the jar model.

3 The Study

The main purpose of this study being undertaken was to investigate the knowledge and understanding of students in Form 1 classes in one government school in Brunei Darussalam on the topic of integers and to investigate if the 'jar model' enhance students' understanding in the topic of integers.

3.1 The Research Questions

The following research questions guide the study:

1. What pre-existing knowledge do the students generally have about integers?
2. To what extent does the strategy used in the intervention enhance the students' performances on operations with integers?

The first research question examined students' prior knowledge including the errors and misconceptions that they hold. These include confusion of rules and instrumental understanding of integers itself.

The second research question examined students' understanding of integers after the teaching of integers and its operations using the 'jar model'.

3.2 Methodology

The present study is an exploratory study which used a multiple perspectives research design. A combination of qualitative and quantitative methods were used to gather data. The sample of the study consisted of Form 1 (grade 7) students in one government secondary school in Brunei. The results of this study presented in this paper were obtained from the analysis of the following data:

1. Document analysis;
2. Analysis of performance data from pencil-and paper pre-test;
3. Interview data analysis – both teachers and students;
4. Analysis of performance data from pencil-and paper post-test;

3.2.1 Pencil and paper test

The Pencil–and–paper test was used to generate pre-test and post-test performance data. The test was administered to all students in July 2006 and September 2006. The Integers test was piloted to test for validity and reliability earlier. There were thirty questions in the Integers Test and they involved questions on each of the four operations of integers (categorised accordingly – positive plus positive, positive minus negative ect.) including the combined operations of integers. Students were categorised into three categories of achievers: high achievers, medium achievers and low achievers.

3.2.2 Interview data Analysis

Class teachers of the classes involved in the study were individually interviewed after the pre-test. The interviews with teachers were audio-taped and analysed. During these interviews, the researcher asked the teachers about the teaching strategies used in their classes, their preferences on which teaching strategies they feel are effective in teaching integers and how they handle students who are still struggling with integers. Each teacher was asked to indicate to what extent most of their students understood the topic integers and on which operations their students had the most problems with.

Twelve students (four high, four medium and four low achievers based on pre-test result), were individually interviewed in July 2006, immediately after the administration of the pre-test. The same twelve students were interviewed again in September 2006 immediately after the post-test. The interviews were tape-recorded. These interviews were conducted to determine the difficulties and the types of errors made by the students. The procedures for interview follows closely the suggestions given by Cohen, Manion and Morrison (2000), to achieve greater validity and to minimise the amount of bias as much as possible.

4 Difficulties in learning integers

The difficulties faced by the students in learning integers are due to the confusion between binary operations of plus and minus and the unary operators which are positive and negative. This confusion is due also to many texts using the same symbols for both plus and positive, and minus and negative. Students always ask 'Why do they have to learn negative numbers and what's the use of negative numbers in our everyday life?' Students often have nothing to relate to, apart from a set of rules governing the combination of negative and positive numbers for the operations. They cannot make sense of the multiplication of a negative number with a negative number and why the product of negative numbers becomes positive.

Teachers also find it easier to teach the rules than to teach for meaning and hope the students' understanding will develop as they operate successfully with the relatively 'simple rules'. Some students find it difficult to establish the rules for themselves; therefore they just rely on remembering them instead of understanding. This can lead to rote learning where students only know how to solve the problems of integers but do not understand why it happens in such a way. Baroody and Ginsburg (1990) described that understanding in mathematics learning involves knowing the concepts and principles related to the procedures being used and making meaningful connections between prior knowledge and the knowledge units being learnt. According to Hart et al (1981) the difficulty is that this stems from the need to work consistently with such rules without recourse to an external, concrete referent and it is this that most secondary school students seem unable to do.

4.1 Some common misunderstandings of integers

Students find integers and operations on integers difficult. The fact that –27 is less than –12 is contrary to the students' experience with (positive) whole numbers. Understanding this requires the students to build mental images and models that allow them to visualize these new comparisons and relationships.

The operation of subtraction, especially subtracting a negative, is difficult for students to make sense of. The idea of subtracting a negative number which gives the same result as adding the opposite of the negative number, is difficult for many students to comprehend. When students have little understanding of subtraction of negative numbers, they may end up just blindly following the rules. Study by Hart et al (1981) found that when students are faced with an expression like +8 – –6 many of them use the rule to work out the appropriate sign and then operate with it (in this case adding 8 and 6) ignoring the starting point. This works in some cases but not in others (such as –2 – –5, where students would give 7 as the answer). This is exactly the kind of error made by the students investigated in this study. Hayes (1999) found that slight misapplications of the rules, such as applying 'two negatives make a positive' to –4 + –2 to get +6, are common and are also common among our students.

The pre-teaching interview data suggested that students responded to integers tasks in a totally mechanical way, with little or no understanding of why they did what they did. Often, students did not know which algorithms they needed to use. Some of the students who did know the algorithms could not identify which algorithm should be associated with which type of problem. It seems that students had not learnt the required concepts and skills properly when they were in primary school and in Form 1 (February 2006).

About 35 per cent of the students made errors when adding two negative numbers together to get a positive. An example would be –2 + (–6) = +8 as identified by Hayes (1999) earlier. About 40 per cent of the students made the error when adding a negative number with a positive number. For example,–2 + 6 = –8 was a common answer, where the students multiplied the negative sign of 2 and the addition operation to get negative and added the numbers. About 41.6 per cent of the students in the study made the error in question –6 + 2 giving the answer as –8 and about 47.7 per cent of the students made an error of –6 + 6 giving the answer as –12. All the errors were made despite the reteaching using the jar model.

For the subtraction of integers, the students in the study also made the same type of error, as the students in the study done by Hart et al (1981) with an expression like 2 – (–6). Many students use the rule to work with the sign, that is, minus and negative become plus and then adding the number. Most of the students ignore the starting sign. It works in this question but not in other questions such as –2 – (–6), where the students would give 8 as the answer. In particular, about 22.1 per cent of the students in the study commited the mistake. About 40.3 per cent of the students made an error of mixing up the rules of addition and subtraction such as –6 – 2 = –4, where students took the sign of 6 since it is larger than 2 and subtracted 6 and 2. About 41.6 per cent of the students made the error when subtracting two negative integers together to get a negative result. An example would be –6 – (–6) = –12. Students knew that when there were two negatives it will become a positive, that is, –6 + 6. Since there is one minus sign the answer is negative. The students were confused with the rule of multiplication, that is positive and negative become negative and added the numbers to get –12. About 34.2 per cent of the students made an error on question 2 – (–6) = –4. Students took only one of the negative signs instead of changing it to become positive.

For the multiplication of integers, about 36.9 per cent of the students made the error to multiply two negative numbers together to get a negative such as question –2 × –2 = –4. The students had misunderstood some part of the rule, that is, the negative sign after multiplication operation was ignored, so –2 × 2 = –4. The same error was made on the division of integers where about 28.8 per cent of the students made the error of dividing two negative numbers to become a negative such as in question –6 ÷ –2 = –3. The students misunderstood the rule as the same in multiplication, that is, a negative sign after a division operation was ignored.

About 35.6 per cent of the students made the error on question 4 – (–2) + 6, where students just take one of the negative signs to make 4 –2 + 6 = 8. This misconception was made by the students in the subtraction of integers. For question (–4 + 6) ÷ –2, about 39.6 per cent of the students made the error of giving the answer as 5. The students got –4 + 6 as –10, then divided it by –2. The students seemed to mix up the rules of addition and multiplication. About

38.9 per cent of the students made the error on question 4 × (–2) – (–6) by giving the answer as 2. The students made the error of 4 × (–2) = 8. Then 8 – (–6) = 2 where the students only took one of the negative signs thinking that they were just the same. The same error was made by the students on the multiplication and subtraction of integers. For question –4 × (2 – 6) ÷ 8, about 45.6 per cent of the students made the error by working out –4 × –4 ÷ 8 = –2. Students made the same error in the multiplication of integers where multiplying two negatives become negative.

5 Results

Performances of the Form 1 students on the pre- and post-teaching test were compared using quantitative procedures. The "test performance" vantage point of Form 1 indicated that the pre-test mean score was 16.50 (out of possible 30) on the integers test. Analyses of the post-teaching data revealed that the mean score of Form 1 students on the integers test was higher than at the pre-teaching stage. The post-test mean score of 21.26 revealed that the students' performances were significantly enhanced. Using the paired t-test, the researcher confirmed that the students' achievements in the five classes are all significantly different. Teaching effects and history was confounded here as well. The post test was given one month after the pre-test. The students were possibly encouraged and motivated to study harder for the test which could have reflected in the improved performances.

Table 1 also shows that all classes scored significantly higher after the intervention using the jar mode.

Class	Post- Test		Pre-Test		Number of Students	t-test	Sig.
	Mean	SD	Mean	SD			
1A	24.92	4.09	20.16	4.60	25	5.170	.000
1B	24.00	4.20	17.58	4.12	36	9.262	.000
1C	21.00	4.44	16.26	4.21	23	4.519	.000
1D	18.25	5.24	15.72	4.79	32	3.009	.005
1E	18.58	4.23	13.48	3.19	33	6.901	.000
Overall	21.26	5.21	16.50	4.67	149	12.46	.000

*$p < .05$

Table 1: Pre and Post-Test Mean Total Score, Standard Deviation, and t-test Results for Students from each of the five classes involved in the study

As for the qualitative data, most of the 12 students that was interviewed seemed to have a slightly better grasp at the post-teaching stage than at the pre-teaching stage of which rules needed to be used to answer questions. The interviewee also tended to make fewer skills manipulation errors than at the pre-teaching stage. However, at the post-teaching stage some interviewees did not have a firm grasp of the jar model concept especially on multiplication and division of integers using the jar model. Post-teaching interviews revealed that the high achiever students managed to answer almost all the 11 questions asked during the interview compared with the pre-teaching interview. Some interviewees were still confused when to remove or add the positive/negative chips from the jar.

Post-teaching interviews had also shown that most of the medium and low achieving students struggled to remember steps on the jar model that had been taught to them. They could not remember when to remove or add the positive and negative chips from the jar especially the subtraction, multiplication and division of integers using the jar model. That was probably because the students may have had to try to learn too many separate skills by rote. This is similar to data reported in Noridah's (1999), Lim's (2000), Khoo's (2001), Zurina's (2003) and Sarina's (2004) dissertations which under examination pressure, most Bruneian students in secondary schools could not remember which skills should be associated with which problems.

6 Discussion and Conclusion

The jar method seemed to generate a better understanding of operations on integers. However, this model can still be improved because students seemed to be confused with some aspects of the model. The model fails to explain situations when a negative number is multiplied by a negative number and when a positive number is divided by a negative number. In cases like this some other explanation need to be given other model need to be combined with the jar model. However, from the study carried out, we can confidently say that the jar model is less confusing than the number line model and created better understanding in the students compared to the rules and analogies that teachers are fond of using before.

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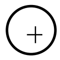
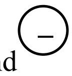
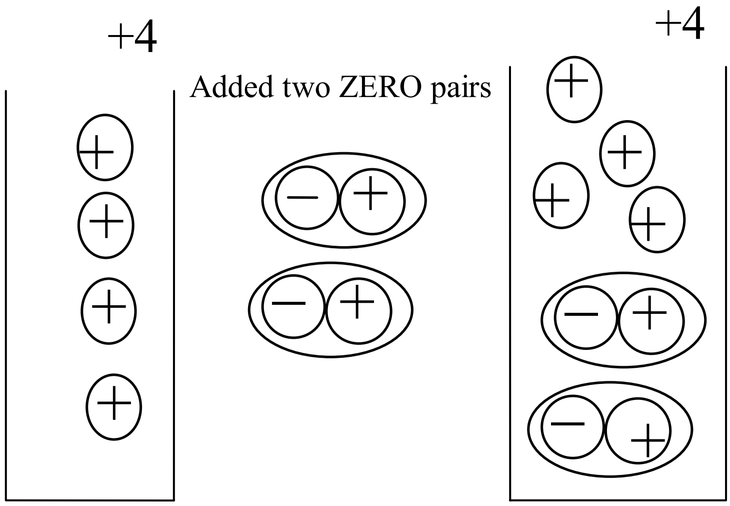
Appendix 1: The Jar Model

The jar model is the model used to teach the students in this study when intervention was implemented. The jar model used positive and negative counters,  and , for students to work with. The same idea had been used by Battista (1983) for teaching integers. Paul Griffith (2002) also advocated a model resembling the jar model. In fact similar models can be found mentioned at many websites (for example, Homeschool Math and Learning Math).

Example: +4 – (–2). You want to remove 2 negative chips from the jar. But there are only 4 positive chips in the jar. To remove 2 negative chips from the jar, we have to add 2 positive/negative pairs into the jar. It is illustrated below:



Now you can take away 2 negative chips and you are left with +6 chips.

$\therefore\ +4 - (-2) = +6$