

Weight Martingale-Ergodic and Ergodic-Martingale Theorems

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Abstract: In this paper we prove weighted martingale-ergodic and weighted ergodic-martingale theorems. Furthermore, analogous dominant and maximal inequalities for weighted martingale ergodic sequences and weighted ergodic martingale averages are also obtained.

Key words: Ergodic averages . martingale . ergodic-martingale . martingale-ergodic averages

INTRODUCTION

General theories unifying ergodic averages and martingales were reported by Kachurovskii [1-3]. Four different variants for theories unifying ergodic averages and martingales have been reported in [4-7]. Besides, one parameter weighted ergodic theorem and multiparameter weighted ergodic theorems have been investigated by Baxter J.H. Olsen [8] and R.L. Jones, J.H. Olsen [9], respectively. In [10], M. Lin and M. Weber considered weighted ergodic theorems and strong laws of large numbers. General ergodic theory is reported in [11].

In this paper we prove weighted martingale-ergodic and weighted ergodic-martingale theorems. Furthermore, analogous dominant and maximal inequalities for weighted martingale ergodic sequences and weighted ergodic martingale averages are also obtained.

Preliminaries: Let $(\Omega, \Sigma, \lambda)$ be a space with a finite measure, $L_0 = L_0(\Omega)$ be a space of complex measurable functions on Ω ,

$$L_p = \{f \in L_0 : \int_{\Omega} |f|^p d\lambda < \infty\},$$

$p \geq 1$, with the norm

$$\|f\|_p = \left(\int_{\Omega} |f|^p d\lambda \right)^{\frac{1}{p}}$$

if $1 \leq p < \infty$, $\|f\|_{\infty} = \sup \{|f(\omega)| : \omega \in \Omega\}$ if $p = \infty$.

Let $\{\mathcal{A}_n\}_{n=1}^{\infty}$ be a monotone sequence of σ -subalgebras of Σ , $\mathcal{A}_n \uparrow \mathcal{A}_{\infty}$ (or $\mathcal{A}_n \downarrow \mathcal{A}_{\infty}$) $E: L_p \rightarrow L_p$ be the expectation operator, $T: L_p \rightarrow L_p$ be the Dunford-Schwartz operator. Put

$$S_n(f, T) = \frac{1}{n} \sum_{k=1}^n T^k f, f^* = \lim_{n \rightarrow \infty} S_n(f, T), f_{\infty}^* = E(f^* | \mathcal{A}_{\infty}),$$

$$S_n(f, T) = \frac{1}{n} \sum_{k=1}^n T^k f, f^* = \lim_{n \rightarrow \infty} S_n(f, T), f_{\infty}^* = E(f^* | \mathcal{A}_{\infty})$$

In [1] it is proved the following

Theorem 1.1

1) Let $f \in L_p, p \in [1, \infty)$. Then $E(S_n(f, T) | \mathcal{A}_n) \rightarrow f_{\infty}^*$ in L_p , in this case $\|f_{\infty}^*\|_p \leq \|f\|_p$.

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2) Let $f \in L_p, p \in (1, \infty)$. Then

$$E(S_n(f, T)|\mathcal{A}_n) \xrightarrow{(o)} f_\infty^* \text{ in } L_p.$$

3) Let $f \in L_1, \sup_{n \geq 1} |S_n(f, T)| \in L_1$. Then $E(S_n(f, T)|\mathcal{A}_n) \xrightarrow{(o)} f_\infty^*$ in L_0 .

Put $f_\infty = E(f|\mathcal{A}_\infty), S_n(f_\infty, T) = \frac{1}{n} \sum_{k=0}^{n-1} T^k f_\infty, f_\infty^* = \lim_{n \rightarrow \infty} S_n(f_\infty, T)$.

Theorem 1.2: [2]

1) Let $f \in L_p, p \in [1, \infty)$. Then $S_n(E(f|\mathcal{A}_n), T) \rightarrow f_\infty^*$ in L_p as $n \rightarrow \infty$.

2) Let $f \in L_p, p \in (1, \infty)$. Then

$$S_n(E(f|\mathcal{A}_n), T) \xrightarrow{(o)} f_\infty^* \text{ in } L_p.$$

3) Let $f \in L_1, \sup |E(f|\mathcal{A}_n)| \in L_1$. Then $S_n(E(f|\mathcal{A}_n), T) \xrightarrow{(o)} f_\infty^*$ in L_0 .

Lemma 1: [2, 12] Let $\{f_n\} \in L_p, p \in [1, \infty)$ and $f_n \rightarrow f^*$ in L_p , as $n \rightarrow \infty$. Then $E(f_n|\mathcal{A}_n) \rightarrow E(f^*|\mathcal{A}_\infty)$ in L_p as $n \rightarrow \infty$.

Lemma 2: [2, 12] Let $f_n \xrightarrow{(o)} f^*$ as $n \rightarrow \infty$ and $\sup_{n \geq 1} |f_n| \in L_1$. Then

$$E(f_n|\mathcal{A}_n) \xrightarrow{(o)} E(f^*|\mathcal{A}_\infty) \text{ in } L_0.$$

Lemma 3: [11] Let $f_n \xrightarrow{(o)} f^*$ as $n \rightarrow \infty$ and $\sup_{n \geq 1} |f_n| \in L_1$. Then $S_n(f_n, T) \xrightarrow{(o)} f^*$ in L_0 as $n \rightarrow \infty$ where $f^* = \lim_n S_n(f, T)$.

Definition 1.3: The sequence numbers $\alpha(k)$ is called Besicovich sequences such that given $\varepsilon > 0$, there is a trigonometric polynomial $\psi_\varepsilon(k)$ such that $\lim_{n \rightarrow \infty} \sup_n \frac{1}{n} \sum_{k=0}^{n-1} |\alpha(k) - \psi_\varepsilon(k)| < \varepsilon$.

A sequence $\alpha(k)$ is called bounded Besicovich if $\{\alpha(k)\} \in l^\infty$. In [9] it was proved the following

Theorem 1.4: Let T be denoted the Dunford-Schwartz operator. Then there exists

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \alpha(k) T^k(f)$$

a.e. for every $f \in L_p, 1 < p < \infty$ and all bounded Besicovich sequences $\alpha(k)$. In this case, if $p \in (1, \infty)$, then $A_n(f, T) = \frac{1}{n} \sum_{k=0}^{n-1} \alpha(k) T^k(f)$ (o)-converges in L_p and if $p = 1$, then $A_n(f, T)$ (o)-converges in L_0 .

In this paper we prove theorems analogous to 1.1 and 1.2 in the case of weighted averages.

WEIGHTED MARTINGALE-ERGODIC THEOREMS

Let $\{\mathcal{A}_n\}_{n=1}^\infty$ be a monotone sequence of σ -subalgebras of $\Sigma, \mathcal{A}_n \uparrow \mathcal{A}_\infty$ (or $\mathcal{A}_n \downarrow \mathcal{A}_\infty$), $\alpha(k)$ be a Besicovich bounded sequence, $T: L_p \rightarrow L_p$ be the Dunford-Schwartz operator. We put,

$$A_n(f, T) = \frac{1}{n} \sum_{k=0}^{n-1} \alpha(k) T^k(f), f^* = \lim_{n \rightarrow \infty} A_n(f, T), f_\infty^* = E(f^*|\mathcal{A}_\infty).$$

Theorem 2.1

1) Let $f \in L_p, p \in (1, \infty)$. Then $E(A_n(f, T)|\mathcal{A}_n) \xrightarrow{(o)} f_\infty^*$ in L_p .

2) If $f \in L_1$ and $\sup_{n \geq 1} A_n(f, T) \in L_1$, then $E(A_n(f, T)|\mathcal{A}_n) \xrightarrow{(o)} f_\infty^*$ in L_0 .

Proof:

1) Since

$$\sup_{n \geq 1} |A_n(f)| = \sup_{n \geq 1} \left| \frac{1}{n} \sum_{k=0}^{n-1} \alpha(k) T^k(f, T) \right| \leq \sup_{n \geq 1} \frac{1}{n} \sum_{k=0}^{n-1} |\alpha(k)| |T|^k(|f|) \leq b \sup_{n \geq 1} \frac{1}{n} \sum_{k=1}^{n-1} |T|^k(f),$$

where $b = \sup |\alpha(k)|$, by the Akcoglu's theorem we have $\sup_{n \geq 1} |A_n(f, T)| \in L_p$.

According to Theorem 1.2 [9], $A_n(f, T) \xrightarrow{(o)} f^*$. Therefore Lemma 2 implies $E(A_n(f, T)|\mathcal{A}_n) \xrightarrow{(o)} E(f^*|\mathcal{A}_\infty) = f_\infty^*$ in L_0 as $n \rightarrow \infty$.

Let $\sup_{n \geq 1} |A_n(f, T)| = h$. Then $E(A_n(f, T)|\mathcal{A}_n) \leq E(h|\mathcal{A}_n)$ for all n . By Theorem 2 [13] $\sup_{n \geq 1} E(h|\mathcal{A}_n) \in L_p$. Hence, $\sup_{n \geq 1} |E(A_n(f, T)|\mathcal{A}_n)| \in L_p$. Therefore $E(A_n(f, T)|\mathcal{A}_n) \xrightarrow{(o)} f_\infty^*$ in L_p .

2) By Theorem 1.4 [9] we have $A_n(f, T) \rightarrow f^*$ a.e. As $\sup_{n \geq 1} A_n(f, T) \in L_1$, by Lemma 2 we obtain $E(A_n(f, T)|\mathcal{A}_n) \rightarrow f_\infty^*$ a.e. Since the convergence a.e. is the (o)-convergence in L_0 , we obtain $E(A_n(f, T)|\mathcal{A}_n) \xrightarrow{(o)} f_\infty^*$ in L_0 .

WEIGHTED ERGODIC-MARTINGALE THEOREM

Let, as in Section 2, $\{\mathcal{A}_n\}_{n=1}^\infty$ be a monotone sequence of σ -subalgebras of Σ , $\mathcal{A}_n \uparrow \mathcal{A}_\infty$ (or $\mathcal{A}_n \downarrow \mathcal{A}_\infty$), $\alpha(k)$ be a Besicovich bounded sequence, $T: L_p \rightarrow L_p$ is the Dunford-Schwartz operator. We put

$$\begin{aligned} f_\infty &= E(f|\mathcal{A}_\infty), \\ A_n(f_\infty, T) &= \frac{1}{n} \sum_{k=0}^{n-1} \alpha(k) T^k(f_\infty), \\ f_\infty^* &= \lim_{n \rightarrow \infty} A_n(f_\infty, T). \end{aligned}$$

Lemma 3.1: Let $f_n \xrightarrow{(o)} f$, $h = \sup_n |f_n| \in L_1$, and $f^* = \lim_{n \rightarrow \infty} A_n(f, T)$. Then $A_n(f_n, T) \xrightarrow{(o)} f^*$ in L_0 .

Proof: Let $h_n = \sup_{m \geq n} |f_m - f|$.

Obviously,

$$|A_n(f_n, T) - f^*| \leq |A_n(f_n, T) - A_n(f, T)| + |A_n(f, T) - f^*|. |A_n(f_n, T) - f^*| \leq |A_n(f_n, T) - A_n(f, T)| + |A_n(f, T) - f^*|.$$

By theorems 1.2 and 1.4 [9], $A_n(f, T) \xrightarrow{(o)} f^*$ as $n \rightarrow \infty$ in L_0 . We will prove that

$$A_n(f_n, T) - A_n(f, T) \xrightarrow{(o)} 0 \text{ as } n \rightarrow \infty \text{ in } L_0.$$

It is clear,

$$\begin{aligned} |A_n(f_n, T) - A_n(f, T)| &= |A_n(f_n - f, T)| \leq \frac{1}{n} \sum_{k=0}^{n-1} |\alpha(k)| |T|^k(|f_n - f|) \leq \\ &b \cdot \frac{1}{n} \sum_{k=0}^{n-1} |T|^k(|f_n - f|) \leq b \cdot S_n(|f_n - f|, |T|). \end{aligned}$$

Since $|f_n - f| \xrightarrow{(o)} 0$ and $\sup_n |f_n - f| \leq 2h \in L_1$, we obtained from Lemma 3, that $S_n(|f_n - f|, |T|) \xrightarrow{(o)} 0$ as $n \rightarrow \infty$. Therefore $A_n(f_n, T) - A_n(f, T) \xrightarrow{(o)} 0$ and $A_n(f_n, T) \xrightarrow{(o)} f^*$ in L_0 .

Theorem 3.2

- 1) Let $p \in (1, \infty)$. Then $A_n(E(f|\mathcal{A}_n), T) \xrightarrow{(o)} f_\infty^*$ in L_p as $n \rightarrow \infty$.
- 2) Let $p = 1$ and $\sup_n |E(f|\mathcal{A}_n)| \leq h \in L_1$. Then $A_n(E(f|\mathcal{A}_n), T) \xrightarrow{(o)} f_\infty^*$ in L_0 .

Proof:

1) According to Theorem 2 [13], $\sup_{n \geq 1} E(|f|\mathcal{A}_n) \in L_p$. Therefore by Lemma 3.1, we have

$$A_n(E(f|\mathcal{A}_n), T) \xrightarrow{(o)} f_\infty^* \text{ in } L_0.$$

Let $\sup_{n \geq 1} E(|f|\mathcal{A}_n) = h$. Then $E(|f|\mathcal{A}_n) \leq h$ and $A_n(E(|f|\mathcal{A}_n), |T|) \leq A_n(h, |T|)$. As by the Akcoglu's theorem $\sup_{n \geq 1} A_n(h, |T|) \in L_p$, we have $\sup_{n \geq 1} A_n(E(|f|\mathcal{A}_n)|T|) \in L_p$ and $\sup_{n \geq 1} |A_n(E(f|\mathcal{A}_n), T)| \in L_p$.

Hence, $A_n(E(f|\mathcal{A}_n)T) \xrightarrow{(o)} f_\infty^*$.

2) By the Theorem 3 [13], we have

$$E(f|\mathcal{A}_n) \xrightarrow{(o)} E(f, \mathcal{A}_\infty)$$

in L_0 . As $\sup_{n \geq 1} |E(f|\mathcal{A}_n)| \in L_1$, by Lemma 3.1 we obtain

$$A_n(E(f|\mathcal{A}_n)T) \xrightarrow{(o)} f_\infty^*$$

in L_0 .

WEIGHTED DOMINANT AND MAXIMAL INEQUALITIES

Theorem 4.1: Let $p \in (1, \infty)$ and $\mathcal{A}_n \downarrow \mathcal{A}_\infty$. Then for $f \in L_p$ the following inequality holds:

$$\| \sup_{n \geq 1} |E(A_n(f, T)|\mathcal{A}_n)| \|_{L_p} \leq bq^2 \| f \|_{L_p},$$

where $b = \sup_k |\alpha(k)|$.

Proof: Let $g = \sup_{n \geq 1} |A_n(f, T)|$. Then

$$|A_n(f, T)| \leq g, \quad \sup_{n \geq 1} |E(A_n(f, T)|\mathcal{A}_n)| \leq \sup_{n \geq 1} E(g|\mathcal{A}_n) |A_n(f, T)| \leq g, \quad \sup_{n \geq 1} |E(A_n(f, T)|\mathcal{A}_n)| \leq \sup_{n \geq 1} E(g|\mathcal{A}_n)$$

and therefore

$$\| \sup_{n \geq 1} |E(A_n(f, T)|\mathcal{A}_n)| \|_{L_p} \leq \| \sup_{n \geq 1} E(g|\mathcal{A}_n) \|_{L_p}.$$

By the dominant inequality for submartingale [12] we have

$$\| \sup_{n \geq 1} E(g|\mathcal{A}_n) \|_{L_p} \leq q \| E(g|\mathcal{A}_1) \|_{L_p}.$$

Since the conditional expectation operator is contracting in L_p we obtain, that

$$\| E(g|\mathcal{A}_1) \|_{L_p} \leq \| g \|_{L_p}.$$

Therefore

$$\| \sup_{n \geq 1} |E(A_n(f, T)|\mathcal{A}_n)| \|_{L_p} \leq q \| \sup_{n \geq 1} |A_n(f, T)| \|_{L_p}.$$

It follows from Akcoglu's theorem that $\| \sup_{n \geq 1} |A_n(f, T)| \|_{L_p} \leq b \cdot q \| f \|_{L_p}$. Thus, for $f \in L_p$ the inequality

$$\| \sup_{n \geq 1} |E(A_n(f, T) | \mathcal{A}_n)| \|_{L_p} \leq bq^2 \| f \|_{L_p}$$

holds.

Theorem 4.2: Let $f \in L_p, p \in (1, \infty)$ and $\mathcal{A}_n \downarrow \mathcal{A}_\infty$. Then for any $\varepsilon > 0$ the following inequality holds

$$\lambda\{\sup_{n \geq 1} |E(A_n(f, T) | \mathcal{A}_n)| \geq \varepsilon\} \leq \frac{1}{\varepsilon^p} \cdot b^p \cdot \| f \|_{L_p}^p,$$

where $b = \sup_k |\alpha(k)|$.

Proof: Let $g = \sup_{n \geq 1} |A_n(f, T)|$. Then

$$\lambda\{\sup |E(A_n(f, T) | \mathcal{A}_n)| \geq \varepsilon\} \leq \lambda\{\sup E(g | \mathcal{A}_n) \geq \varepsilon\}. \lambda\{\sup |E(A_n(f, T) | \mathcal{A}_n)| \geq \varepsilon\} \leq \lambda\{\sup E(g | \mathcal{A}_n) \geq \varepsilon\}.$$

By the maximal inequality for for submartingale $E(g | \mathcal{A}_n)$ [12] we have, that

$$\lambda\{E(g | \mathcal{A}_n) \geq \varepsilon\} \leq \frac{1}{\varepsilon^p} \| E(g | \mathcal{A}_1) \|_{L_p}^p.$$

Applying the contracting property of the conditional expectation operator in L_p , we obtain

$$\| E(g | \mathcal{A}_n) \|_{L_p}^p \leq \| g \|_{L_p}^p = \| \sup_{n \geq 1} |A_n(f, T)| \|_{L_p}^p \leq \| \sup_{n \geq 1} A(|f|, |T|) \|_{L_p}^p.$$

Now applying the inequality $\| \sup_{n \geq 1} A_n(|f|, |T|) \|_{L_p} \leq b \cdot q \| f \|_{L_p}$, we have

$$\lambda\{\sup_{n \geq 1} |E(A_n(f, T) | \mathcal{A}_n)| \geq \varepsilon\} \leq \frac{1}{\varepsilon^p} b^p \cdot q^p \cdot \| f \|_{L_p}^p \cdot b = \sup_k |\alpha(k)|.$$

The following theorems can be proved analogously to theorems 4.1 and 4.2.

Theorem 4.3: Let $p \in (1, \infty)$. Then for $f \in L_p$ the following inequality holds

$$\| \sup_{n \geq 1} |A_n(E(f | \mathcal{A}_n) T)| \|_{L_p} \leq b \cdot q^2 \| f \|_{L_p},$$

where $b = \sup_k |\alpha(k)|$.

Theorem 4.4: Let $p \in (1, \infty)$. Then for any $\varepsilon > 0$ and $f \in L_p$ the following inequality holds

$$\lambda\{\sup_{n \geq 1} |A_n(E(f | \mathcal{A}_n), T)| \geq \varepsilon\} \leq \frac{1}{\varepsilon^p} b^p \cdot q^p \| f \|_{L_p}^p,$$

where $b = \sup_k |\alpha(k)|$.

MULTIPARAMETER WEIGHTED MARTINGALE ERGODIC THEOREMS

In this section we define the d -dimensional case of martingale ergodic averages and consider convergence theorems for such averages.

Let $\{a(k) : k \in Z_d^+\}$ be the class of weights and

$$k = (k_1, k_2, \dots, k_d), \mathbf{N} = (N_1, N_2, \dots, N_d), |\mathbf{N}| = N_1 \cdot N_2 \cdot \dots \cdot N_d, \mathbf{0} = (0, 0, \dots, 0). k = (k_1, k_2, \dots, k_d), \mathbf{N} = (N_1, N_2, \dots, N_d), |\mathbf{N}| = N_1 \cdot N_2 \cdot \dots \cdot N_d, \mathbf{0} = (0, 0, \dots, 0).$$

For $N = (N_1, N_2, \dots, N_d)$ $N \rightarrow \infty$ means that $N_i \rightarrow \infty$ for any $i = 1, 2, \dots, d$.

Definition 5.1: [9] The sequence $\{a(k): k \in Z_d^+\}$ is called r -Besicovich if for every $\varepsilon > 0$ there is a sequence of trigonometric polynomials in d variables, $\psi_\varepsilon(k)$, such that $\lim_{|N| \rightarrow \infty} \sup \frac{1}{|N|} \sum_{k=0}^{N-1} |a(k) - \psi_\varepsilon(k)| < \varepsilon$.

This class is denoted by $B(r)$. The sequence $\alpha(k)$ is called r -bounded Besicovitch if $\{\alpha(k)\} \in B(r) \cap l^\infty$. If $\{\alpha(k)\} \in B(1) \cap l^\infty$ then $\{\alpha(k)\}$ is called a bounded Besicovich sequence.

Let T_1, T_1, \dots, T_1 denote a family of d linear operators in L_p . We consider averages

$$A_N(\mathbf{T})f = \frac{1}{|N|} \sum_{k=0}^N a(k) \mathbf{T}^k f,$$

where $f \in L_p, \mathbf{T}^k = T_1^{k_1} \cdot T_2^{k_2} \cdot \dots \cdot T_d^{k_d}$

In [9] it is proved the following

Theorem 5.2: Let $T = (T_1, T_1, \dots, T_1)$ denoted d Dunford-Schwartz operators. Then $A_N(\mathbf{T})f$ converges a.e. for every $f \in L_p, 1 < p \leq \infty$ and all bounded Besicovich sequences $\alpha(k)$.

Let, as in the Section 2 $\{\mathcal{A}_n\}_{n=1}^\infty$ be a monotone sequence of σ -subalgebras of Σ and $\mathcal{A}_n \downarrow \mathcal{A}_\infty$. We put $f^* = \lim_{N \rightarrow \infty} A_N(\mathbf{T})f, f_\infty^* = E(f^* | \mathcal{A}_\infty)$.

Theorem 5.3: Let $f \in L_p, 1 < p \leq \infty, \alpha(k)$ be a bounded Besicovich sequence, $N_n = (N_1^n, N_2^n, \dots, N_d^n) \rightarrow \infty$ as $n \rightarrow \infty$. Then

$$E(A_{N_n}(\mathbf{T})f | \mathcal{A}_n) \xrightarrow{(o)} f^*$$

in L_p .

The proof of Theorem 5.3 is analogous to the proof of Theorem 2.1 1).

Theorem 5.4: Let $\mathcal{A}_l \downarrow \mathcal{A}_\infty, f \in L_p, 1 < p \leq \infty, \alpha(k)$ bounded Besicovich sequence, $b = \sup_k |\alpha(k)|, N_l = (N_1^l, N_2^l, \dots, N_d^l) \rightarrow \infty$ as $l \rightarrow \infty$. Then

- 1) $\| \sup_l |E(A_{N_l}(\mathbf{T})f | \mathcal{A}_l)| \|_p \leq b \cdot q^{d+1} \|f\|_p;$
- 2) $\lambda\{\sup_l |E(A_{N_l}(\mathbf{T})f | \mathcal{A}_l)| \geq \varepsilon\} \leq b \cdot q^{pd} \frac{\|f\|_p^p}{\varepsilon^p}.$

Proof

- 1) Let $g_l = \sup_{m \geq l} |A_{N_m}(\mathbf{T})f|$. Then

$$|E(A_{N_l}(\mathbf{T})f | \mathcal{A}_l)| \leq E(\sup_{m \geq l} |A_{N_m}(\mathbf{T})f | \mathcal{A}_l) = E(g_l | \mathcal{A}_l)$$

and

$$\sup_l |E(A_{N_l}(\mathbf{T})f | \mathcal{A}_l)| \leq \sup_l E(g_l | \mathcal{A}_l). \tag{1}$$

Therefore

$$\| \sup_l |E(A_{N_l}(\mathbf{T})f | \mathcal{A}_l)| \|_p \leq \| \sup_l E(g_l | \mathcal{A}_l) \|_p.$$

By the dominant inequality,

$$\| \sup_l E(g_l | \mathcal{A}_l) \|_p \leq q \| E(g_1 | \mathcal{A}_1) \|_p.$$

Since the conditional expatiation operators is contracting in L_p , we have

$$\| E(g_1 | \mathcal{A}_1) \|_p \leq \| g_1 \|_p. \tag{2}$$

Thus

$$\begin{aligned} & \| \sup_l | E(A_{N_l}(\mathbf{T})f | \mathcal{A}_l) | \|_p \leq q \| g_1 \|_p = \\ & = q \| \sup_m A_{N_m}(\mathbf{T})|f| \|_p \leq q \cdot q^d \cdot b \| f \|_p = \\ & = b \cdot q^{d+1} \| f \|_p. \end{aligned}$$

2) By the inequality (1) we have

$$\lambda\{\sup_l | E(A_{N_l}(\mathbf{T})f | \mathcal{A}_l) | \geq \varepsilon\} \leq \lambda\{\sup_l E(g_l | \mathcal{A}_l) \geq \varepsilon\} \cdot \lambda\{\sup_l | E(A_{N_l}(\mathbf{T})f | \mathcal{A}_l) | \geq \varepsilon\} \leq \lambda\{\sup_l E(g_l | \mathcal{A}_l) \geq \varepsilon\}.$$

According to the maximal inequality

$$\lambda\{\sup_l E(g_l | \mathcal{A}_l) \geq \varepsilon\} \leq \frac{1}{\varepsilon^p} \| E(g_l | \mathcal{A}_l) \|_p^p.$$

Applying inequality (2), we obtain

$$\frac{1}{\varepsilon^p} \| E(g_l | \mathcal{A}_l) \|_p^p \leq \frac{1}{\varepsilon^p} \| g_1 \|_p^p \leq \frac{1}{\varepsilon^p} \| \sup_m A_{N_m}(\mathbf{T})|f| \|_p^p.$$

Since $\| \sup_m A_{N_m}(\mathbf{T})|f| \|_p^p \leq b^p q^{dp} \| f \|_p^p$, we obtain

$$\lambda\{\sup_l | E(A_{N_l}(\mathbf{T})f | \mathcal{A}_l) | \geq \varepsilon\} \leq b^p \cdot q^{pd} \frac{\| f \|_p^p}{\varepsilon^p}.$$

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