A RELIABLE MODIFICATION OF THE HOMOTOPY PERTURBATION METHOD FOR DIFFERENTIAL AND INTEGRAL EQUATIONS

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Abstract—*In this paper, a new reliable modification of the homotopy perturbation method (HPM) is introduced and applied to the differential and integral equations. A comparative study between the modified homotopy method (MHPM) and the standard HPM is conducted. The efficiency of the modified technique is examined by several illustrative examples. In all cases of differential and integral equations, the modified HPM yields the exact solutions in two iterations only.*

Keywords**:** *Homotopy-perturbation method, differential equations, integral equations*

I. INTRODUCTION

There exists a wide body of literature dealing with the problem in frontier science and engineering is the physically correct solution of linear or nonlinear problems modeled by ordinary differential equations (ODEs) or partial differential equations (PDEs) subject to general initial or boundary conditions. Finding accurate and efficient methods for solving nonlinear ODEs or PDEs has long been an active research undertaking. Recently, Wazwaz [1] proposed a new modification of the Adomian decomposition method (ADM) to handle the ODEs and integral differential equations. Very recently, Belal et al. [2] obtained exact solutions of the nonlinear systems of PDEs studied directly via VIM.

In recent years, much attention has been devoted to the study of the homotopy-perturbation method (HPM) [3-10] for solving a wide range of problems whose mathematical models yield differential equation or system of differential equations. HPM deforms a difficult problem into a set of problems which are easier to solve without any need to transform nonlinear terms. The applications of HPM in nonlinear problems have been demonstrated by many researchers, cf. [11-14]. Recently, HPM was employed for solving singular second-order differential equations [15], nonlinear population dynamics models [16] and time-dependent Emden-Fowler type equations [17], the Klein-Gordon and sine-Gordon equations [18]. Very recently, Chowdhury et al. [19] were the first to successfully apply the multistage homotopy-perturbation method (MHPM) to the chaotic Lorenz system and Odibat [20] propose a new modification of the HPM for linear and nonlinear operators.

The aim of this work is to present an alternative approach called modified HPM based on standard HPM for finding series solutions to linear and nonlinear differential and integral equations. The efficiency and accuracy of HPM and modified HPM are demonstrated through several test examples.

II. MODIFIED TECHNOQUE DESCRIPTION

Homotopy-pertuebation method (HPM) is a novel and effective method, and can solve various nonlinear problems. The basic ideas of this method can be found some of He's papers [7-14]. In this section, we shall introduce a new reliable procedure for choosing the initial approximations in HPM to handle linear, nonlinear inhomogeneous differential equations and integral equations. To do so, we consider the following general nonlinear differential equation

$$\mathrm{Lu} + \mathrm{Ru} + \mathrm{Nu} = \mathrm{g}(x), \tag{1}$$

where L is the highest order derivative which is assumed to be easily invertible, R the linear differential operator of order less than L, Nu represents the nonlinear terms, and g is the source term.

According to the HPM, we construct a homotopy of Eq. (1) which satisfies

$$\begin{aligned} H(u,p) = L(u) - L(v_0) + pL(v_0) \\ + p[R(u) + N(u) - g(x)] = 0, \end{aligned} \tag{2}$$

where $p \in [0,1]$ is an embedding parameter and $u_0 = v_0$ is an initial approximation which satisfies boundary conditions. When we put $p = 0$ and $p = 1$ in Eq. (2), we get

$$H(u,0) = L(u) - L(v_0) = 0 \quad \text{and} \qquad (3)$$
$$H(u,1) = Lu + Ru + Nu - g(x) = 0,$$

which are the linear and nonlinear original equations respectively. In topology this called deformation and $L(u) - L(u_0)$) and $\mathrm{Lu} + \mathrm{Ru} + \mathrm{Nu} - \mathrm{g(x)}$ are called homotopic. Supposing the solution of (1) can be expressed as

$$u(x) = \sum_{n=0}^{\infty} p^n u_n = u_0(x) + pu_1(x) + p^2 u_2(x) + p^3 u_3(x) + \cdots. \qquad (4)$$

According to HPM, the approximate solution of Eq. (4) can be expressed as a series of the power of *p*, i.e.,

$$u = \lim_{p \to 1} u = u_0 + u_1 + u_2 + u_3 \cdots. \qquad (5)$$

The series (5) is convergent in most of the cases. However the rate of convergence depends on *L(u)* [9].

Now we substitute (4) into (2) and equating the like terms of *p,* we obtain

$$u_0(x) = L^{-1}(g(x)) + \phi(x) = f(x)$$
$$p^{k+1} \ : \ u_{k+1}(x) = -L^{-1}(Ru_k) - L^{-1}(Nu_k) = -L^{-1}(Ru_k) - L^{-1}(H_k), \ k \ge 0, \qquad (6)$$

where the function $f(x)$ represents the terms arising from integrating the source term $g(x)$ and from using the given conditions, $\phi(x)$, all of which are assumed to be prescribed.

The nonlinear term $\mathrm{Nu_k} = \mathrm{F(u)}$ is usually represented by an infinite series of the so-called He's polynomials {Gorbani},

$$F(u) = \sum_{k=0}^{\infty} H_k \ .$$

The polynomials H_k are generated for all kinds of nonlinearity so that A_0 depends only on u_0, A_1 depends on u_0 and u_1, and so on. The He's polynomial $A_k(u_0, u_1, u_2, \cdots, u_k)$ [21], is given by,

$$H_k = \frac{1}{k!} \frac{d^k}{dp^k} \left[N\left(\sum_{i=0}^{k} p^i u_i \right) \right]_{p=0}$$

The modification: Inspired by Wazwaz [1], we introduce an alternative way of choosing the initial approximations, that is

$$\mathrm{v_0} = \mathrm{L}^{-1}(\mathrm{g(x)}) + \phi(\mathrm{x}) = \mathrm{f(x)} \ . \qquad (7)$$

The modified form is based on the assumption that the initial approximation $\mathrm{v_0}$ given in Eq. (7) can be decomposed into two parts, namely $\mathrm{f_0}$ and $\mathrm{f_1}$ such that

$$\mathrm{f} = \mathrm{f_0} + \mathrm{f_1}.$$

Based on this, we suggest slight variation in the standard HPM on the components $\mathrm{u_0}$ and $\mathrm{u_1}$. The suggestion is that only the part $\mathrm{f_0}$ be combined with the component $\mathrm{u_0}$ and $\mathrm{f_1}$ be added with $\mathrm{u_1}$. Under this assumption we obtained Eqn. (6) as follows

$$u_0(x) = f_0(x) \qquad (3)$$
$$p^1 : \ u_1 = f_1 - L^{-1}(Ru_0) - L^{-1}(Nu_0)$$
$$p^{k+2} : u_{k+2}(x) = -L^{-1}(Ru_{k+1}) - L^{-1}(Nu_{k+1}) = -L^{-1}(Ru_{k+1}) - L^{-1}(H_{k+1}), \ k \ge 0. \qquad (8)$$

We show that the zeroth component u_o in the recursive scheme of the standard HPM (6) is defined by the total function $f(x)$, but in recursive scheme (8) of the modified HPM the zeroth component u_o is defined only by a part $f_0(x)$ of $f(x)$. And the remaining part $f_1(x)$ of $f(x)$ is added to the component u_1 in (8). The small difference of reducing the number of terms of $\mathrm{u_0}$ could reduce the computational work. Furthermore, because of the dependence of the He's polynomials on the initial component $\mathrm{u_0}$ in the nonlinear equations, the reduction of terms in $\mathrm{u_0}$ could reduce calculations. Additional, this small difference in the components $\mathrm{u_0}$ and $\mathrm{u_1}$ may give the exact solution by using two iterations only. However, the success of the MHPM depends completely on the correct selection of the function f_0 and f_1, here the trials are the only technique that can be used.

III. APPLICATIONS

In order to assess both the applicability and the accuracy of the modified procedure described above, some test examples are considered for linear, nonlinear inhomogeneous differential equations and integral equations.

Example 1

First we consider the ordinary differential equation

$$\frac{\mathrm{du}}{\mathrm{dx}} - \mathrm{u} = \mathrm{x}\cos\mathrm{x} - \mathrm{x}\sin\mathrm{x} + \sin\mathrm{x}, \quad \mathrm{u}(0) = 0 \qquad (7)$$

Standard HPM: We construct a homotopy which satisfies the following relation

$$\frac{du(x)}{dx} - \frac{dv_0}{dx} + p(\frac{dv_0}{dx} - u(x) - x\cos x + x\sin x - \sin x) = 0.$$

The iterative formula based on (6) is given by

$$u_0(x) = \int_0^x (x\cos x - x\sin x + \sin x)dx = x\sin x + x\cos x - \sin x$$
$$u_1(x) = \int_0^x (x\sin x + x\cos x - \sin x)dx = -x\cos x + \sin x + x\sin x + 2\cos x - 2$$

$$u_2(x)=\int_0^x(-x\cos x+\sin x+x\sin x+2\cos x-2)dx$$
$$=-x\sin x-2\cos x-x\cos x+3\sin x-2x+2,$$
$$u_3(x)=\int_0^x(-x\sin x-2\cos x-x\cos x+3\sin x-2x+2)dx=4+2x-x^2-3\sin x+x\cos x-4\cos x-x\sin x,$$
$$u_4(x)=\int_0^x(4+2x-x^2-3\sin x+x\cos x-4\cos x-x\sin x)dx$$
$$=-4+4x+x^2-\frac{1}{3}x^3-5\sin x+x\cos x+4\cos x+x\sin x,$$

etc. Hence the 4-term approximate series solution is

$$u(x)=-\frac{1}{3}x^3+4x-5\sin x+x\cos x+x\sin x$$

and this will be needed more terms to yield the close-form solution. However, we see that the noise terms $x\cos x$ and $-\sin x$ in u_0 will appear in u_1 with opposite signs and the remaining noise terms in u_1 will be appeared also in u_2 with opposite signs and so on. Therefore proceeding in this way by cancelling these noise terms from series solution and this will in the limit of infinitely many terms gives remaining non-canceled term of u_0 as close-form solution. i.e.

$$u_0=x\sin x.$$

From the above solution it is obvious that the noise terms made a remarkable effect in the convergence of the solution.

The modified HPM: To apply the modified HPM, let us take, $v_0=f=f_0+f_1$, where $f_0=x\sin x$ and $f_1=x\cos x-\sin x$.

The iterative formula based on (8) we obtain,

$$u_0(x)=x\sin x,$$
$$p^1:\ u_1(x)=x\cos x-\sin x-\int_0^x x\sin x\,dx=x\cos x-\sin x-x\cos x+\sin x=0, \tag{8}$$
$$p^{k+2}\ :\ u_{k+2}(x)=0,\ k\geq 0.$$

Hence, by using only two iterations the exact solution is reached,

$u(x)=x\sin x.$

Example 2

Consider the nonhomogeneous advection partial differential equations [1]

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=x+xt^2,\quad u(x,0)=0. \tag{9}$$

Standard HPM: Now we construct a homotopy which satisfies the following relation:

$$\frac{\partial u}{\partial t}-\frac{\partial v_0}{\partial t}+p\left[\frac{\partial v_0}{\partial t}+u\frac{\partial u}{\partial x}-x-xt^2\right]=0.$$

According to iterative formula (6), we obtain

$$u_0(x,t)=\int_0^t(x+xt^2)dt=xt+\frac{1}{3}xt^3,$$
$$u_1(x,t)=-\int_0^t\left(u_0(u_0)_x\right)dt=-\frac{1}{2}\int_0^t\left(u_0^2\right)_x dt=-\frac{1}{63}\mathrm{xt}^7-\frac{2}{15}\mathrm{xt}^5-\frac{1}{3}\mathrm{xt}^3$$
$$u_2(x,t)=-\frac{1}{2}\int_0^t\left(u_1^2\right)_x dt$$
$$=-\frac{1}{59535}xt^5-\frac{4}{12285}xt^{13}-\frac{134}{51975}xt^{11}-\frac{4}{405}xt^9-\frac{1}{63}xt^7,$$

and so on. This will be needed more terms to yields the close-form solutions.

However, we see that the noise terms $\frac{1}{3}xt^3$ in u_0 will appear in u_1 with opposite signs and so on. Therefore proceeding in this way by cancelling these noise terms from series solution and this will in the limit of infinitely many terms gives remaining non-canceled term of u_0 as close-form solution. i.e. $u(x,t)=xt$ with non-canceled noise terms.

The modified HPM: To apply the modified HPM, let us take, $v_0=f=f_0+f_1$, where $f_0=xt$ and $f_1=\frac{1}{3}xt^3$.

The iterative formula based on (8) we obtain,

$$u_0(x,t)=xt,$$
$$p^1:\ u_1(x,t)=\frac{1}{3}xt^3-\frac{1}{2}\int_0^t\left(u_0^2\right)_x dt=0,$$
$$p^{k+2}\ :\ u_{k+2}(x,t)=0,\ k\geq 0.$$

Hence, by using only two iterations the exact solution is reached,

$$u(x,t)=xt.$$

Example 3

Finally, we consider the nonhomogeneous Fredholm integral equation

$$u(x)=\cos^{-1}x-x+\int_0^1 xu(t)dt.$$

Standard HPM: Now we construct a homotopy which satisfies the following relation:

$$u-v_0+p\left[v_0-\cos^{-1}x+x+\int_0^1 xu(t)\,dt\right]=0.$$

According to iterative formula (6), we obtain

$$u_0(x)=v_0=\cos^{-1}x-x,$$
$$u_1(x)=\int_0^t xu_0(t)\,dt=\int_0^t x\left(\cos^{-1}t-t\right)dt=\frac{1}{2}x,$$

$$u_2(x) = \int_0^t x u_1(t)\,dt = \frac{1}{4}x,$$

$$u_3(x) = \int_0^t x u_2(t)\,dt = \frac{1}{8}x,$$

$$u_4(x) = \int_0^t x u_3(t)\,dt = \frac{1}{16}x,$$

$$u_5(x) = \int_0^t x u_4(t)\,dt = \frac{1}{32}x,$$

and so on.

Hence the series solution can be written as

$$u(x) = \cos^{-1} x - x + \frac{1}{2}x\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots\right).$$

The modified HPM: To apply the modified HPM, let us take, $v_0 = f = f_0 + f_1$, where $f_0 = \cos^{-1} x$ and $f_1 = -x$.

The iterative formula based on (8) we obtain,

$$u_0(x) = \cos^{-1} x,$$

$$p^1 : \; u_1(x) = -x + x\int_0^1 u_0(t)\,dt = 0,$$

$$p^{k+2} \; : \; u_{k+2}(x) = 0, \; k \geq 0.$$

Hence, by using only two iterations the exact solution is reached,

$$u(x) = \cos^{-1} x.$$

IV. CONCLUSION

In this paper, we first proposed a reliable modification to the homotopy-perturbation method (HPM) by introducing a new technique to choose initial component that already reduced the computational work and accelerates the rapid convergence of the HPM series solution. We have chosen two examples from differential equations and one example from integral equations. From the test examples, we see that the new modification of the HPM provided exact solution by using only two terms in series solution. However in the standard HPM the noise terms will appear in the series solutions and will be needed infinitely many terms to get the close form solution. It can be concluded that the new modification of HPM is a promising tool for solving linear-nonlinear differential and integral equations.

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