On the numerical solution of linear stiff IVPs by modified homotopy perturbation method

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Abstract. In this paper, we introduce a method to solve linear stiff IVPs. The suggested method, which we call modified homotopy perturbation method, can be considered as an extension of the homotopy perturbation method (HPM) which is very efficient in solving a varety of differential and algebraic equations. In this work, a class of linear stiff initial value problems (IVPs) are solved by the classical homotopy perturbation method (HPM), modified homotopy perturbation method and an explicit Runge-Kutta-type method (RK). Numerical comparisons demonstrate the limitations of HPM and promising capability of the MHPM for solving stiff IVPs. The results prove that the modified HPM is a powerful tool for the solution of linear stiff IVPs.

1 Intruduction

Homotopy perturbation method [1–6] is an analytical method which can be applied to the solution of linear, nonlinear deterministic and stochastic operator equations. HPM deforms a difficult problem into an infinite set of problems which are easier to solve without any need to transform nonlinear terms. The applications of HPM in nonlinear problems have been demonstrated by many researchers. In recent years, much attention has been devoted to the application of the HPM, to the solutions of various scientific models[8–11]. HPM yields rapidly convergent series solutions [12]. Very recently, Chowdhury *et al.* [13], Chowdhury and Hashim [14],Hashim and Chowdhury [15] and Hashim *et al.* [16] were the first to successfully apply the multistage homotopy-perturbation method (MHPM) to the chaotic Lorenz system, Chen system and a class of systems of ODEs. The mathematical equations modelling many real-world physical phenomena are often stiff equations, *i.e.* equations with a wide range of temporal scales. The numerical methods for solving stiff equations must have good accuracy and wide region of stability. Hojjati *et al.* [17] developed a multistep method

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for solving stiff systems of initial value problems (IVPs). Knowing that the classical explicit fourth-order Runge-Kutta method is insufficient for the solution of stiff IVPs, Ahmad *et al.* [18] presented an explicit Taylor-like method for solving stiff IVPs. In Ahmad and Yaacob [19], an explicit Runge-Kutta-like method is developed and shown to be efficient for the solution of stiff ODEs. Very recently, Nie *et al.* [20] presented a class of efficient semi-implicit schemes for stiff reaction-diffusion equations. A variable-step size algorithm for stiff systems has been proposed recently by Jannelli and Fazio [21]. In [22], a class of methods having properties very close to those of traditional Runge-Kutta methods were developed. Butcher and Hojjati [23] devised a class of second derivative methods possessing Runge-Kutta stability property. Hojjati *et al.* [24] presented a new class of second derivative multistep methods with improved stability region.

All of the methods mentioned above need some sort of discretizations. In this work, the HPM was treated as an algorithm for approximating the solutions in a sequence of time intervals (*i.e.* time step). We call this approach as modified HPM (MHPM). In this task, we shall apply the MHPM for the first time to the solutions of stiff linear IVPs. Comparisons will be made against the classical HPM and an explicit Runge-Kutta method to determine the performance of MHPM.

2 Solution approaches

In this section, the HPM is modified in order to obtain the approximate solutions for solving stiff initial value problems:

$$y' = f(t, y) \quad \text{with} \quad y(0) = y_0, \tag{1}$$

where $f(t, y)$ may be a linear or non-linear function.

Now we first write Eq. (1) in the operator form

$$Ly = f(t, y), \tag{2}$$

where $L = \mathrm{d}/\mathrm{d}t$ is easily invertible.

According to HPM, we construct a homotopy into Eq. (2) which satisfies the following relation

$$Ly - Lx_0 + p[Lx_0 - f(t, y)] = 0, \tag{3}$$

where $p \in [0, 1]$ is an embedding parameter and x_0 is an initial approximation which generally satisfies the initial conditions. It is obvious that when the homotopy parameter $p = 0$, Eq. (3) becomes a linear equation and when $p = 1$ we get the original nonlinear equation. According to HPM, the solution form of (1) is written as

$$y(t) = u_0(t) + pu_1(t) + p^2 u_2(t) + \cdots, \tag{4}$$

where $u_j(j = 0, 1, 2, 3, \ldots)$ are functions yet to be determined. Substituting (4) into (3) and collecting terms of the same powers of p, we have

$$Lu_0 - Lx_0 = 0, \quad u_0(t_0) = y_0 \tag{5}$$

$$Lu_1 + Lx_0 - f(t, u_0) = 0, \quad u_1(t_0) = 0, \tag{6}$$

$$Lu_2 - f(t, u_1) = 0, \quad u_2(t_0) = 0, \tag{7}$$

etc. Now we can easily solve the above equations for u_1, u_2, and u_3 etc. using the Maple package. Finally, the series solution can be written as

$$y \simeq u_0 + u_1 + u_2 + u_3 + \cdots. \tag{8}$$

The convergence of series (8) has been proven by He in his papers [2].

Now, we treat the HPM as an algorithm for approximating the dynamical response in a sequence of time intervals (*i.e.* time step) $[0, t_1)$, $[t_1, t_2)$, $\ldots$, $[t_{m-1}, T)$ such that the initial condition in $[t^*, t_{m+1})$ is taken to be the condition at t^*.

3 Numerical application

In this section, we shall demonstrate how well the MHPM compares with the Runge-Kutta-like method of [19] for the solutions of linear stiff IVPs. The HPM iterative algorithm is coded in the computer algebra package Maple. The Maple environment variable `Digits` controlling the number of significant digits is set to 16 in all the calculations done in this paper.

3.1 Problem 1

First, we consider the simple linear equation,

$$\frac{dy}{dt} = -30y \quad \text{with} \quad y(0) = \frac{1}{3}, \tag{9}$$

whose exact solution is given by

$$y = \frac{1}{3}\mathrm{e}^{-30t}. \tag{10}$$

According to HPM, we construct a homotopy into Eq. (9) which satisfies the following relation

$$\frac{dy}{dt} - \frac{dx_0}{dt} + p[\frac{dx_0}{dt} + 30y] = 0, \tag{11}$$

where $p \in [0, 1]$ is an embedding parameter and x_0 is an initial approximation which generally satisfies the initial conditions. It is obvious that when the

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homotopy parameter $p=0$, Eq. (11) becomes a linear equation and when $p=1$ we get the original nonlinear equation. According to HPM, the solution form of (9) is written as

$$y(t) = u_0(t) + pu_1(t) + p^2u_2(t) + \cdots, \tag{12}$$

where $u_j(j = 0, 1, 2, 3, \ldots)$ are functions yet to be determined. Let us consider the initial approximation as

$$x_0(t) = u_0(t) = \frac{1}{3}. \tag{13}$$

Substituting the series solution (12) and (13) into (11) and collecting terms of the same powers of p, we have

$$\frac{du_1}{dt} + 30u_0 = 0, \quad u_1(0) = 0, \tag{14}$$
$$\frac{du_2}{dt} + 30u_1 = 0, \quad u_2(0) = 0, \tag{15}$$
$$\frac{du_3}{dt} + 30u_2 = 0, \quad u_3(0) = 0, \tag{16}$$
$$\frac{du_4}{dt} + 30u_3 = 0, \quad u_4(0) = 0, \tag{17}$$

etc.

Solving the differential equations (14)–(17) we obtain,

$$u_0(t) = \frac{1}{3}, u_1(t) = -10t, u_2(,t) = 150t^2, u_3(t) = -1500t^3, u_3(t) = 11250t^4$$

etc.

The 5-term HPM solution on the first subinterval is easily obtained and given as

$$\phi_5 = \frac{1}{3} - 10t + 150t^2 - 1500t^3 + 11250t^4. \tag{18}$$

This HPM series solution fails to produce reasonable results as shown in the third column of Table 1. However, the HPM solutions can be improved if we treat the classic HPM as an algorithm for approximating the solutions of the stiff equation in a sequence of time intervals (i.e. time steps). From the results presented in Table 1 we see that the MHPM at the time step $h = 10^{-3}$ produces better solutions compared to that obtained by the Runge-Kutta-like method [19].

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Table 1. MHPM solutions using 5 terms as compared with the exact solutions, the classical HPM solutions and solutions from the explicit Runge-Kutta-like method [19] for example 1.

t	Exact, (10)	HPM, ϕ_5	RK [19], $h = 10^{-3}$	MHPM, ϕ_5, $h = 10^{-3}$
0.1	1.65956895E-02	4.58333333E-01	1.66028745E-02	1.65956907E-02
0.2	8.26250726E-04	1.03333333E+01	8.26966326E-04	8.26250050E-04
0.3	4.11366014E-05	6.14583333E+01	4.11900544E-05	4.11365300E-05
0.4	2.04807078E-06	2.12333333E+02	2.05161992E-06	2.04807700E-06
0.5	1.01967440E-07	5.48458333E+02	1.02188364E-07	1.01964600E-07
0.6	5.07665991E-09	1.18233333E+03	5.08986175E-09	5.07588000E-09
0.7	2.52752014E-10	2.25345833E+03	2.53519008E-10	2.52645000E-10
0.8	1.25837818E-11	3.92833333E+03	1.26274328E-11	1.25801000E-11
0.9	6.26509606E-13	6.40045833E+03	6.28955048E-13	6.26590000E-13
1.0	3.11920766E-14	9.890333333E+03	3.13273852E-14	3.11940000E-14

3.2 Problem 2

Finally, we consider the linear nonhomogeneous initial value problem considered in [19],

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -100y + \mathrm{e}^{-2t} \quad \text{with} \quad y(0) = 0. \tag{19}$$

The exact solution is

$$y = \frac{1}{98}\mathrm{e}^{-100t}(-1 + \mathrm{e}^{98t}). \tag{20}$$

By the same manipilations as in the previous example, the 6-term HPM solution on the first subinterval is easily obtained and given as

$$\begin{aligned}\phi_6 = &-156250000\mathrm{e}^{-2t} - \frac{125000000}{3}t^5 + \frac{312500000}{3}t^4 \\ &- \frac{625000000}{3}t^3 + 312500000t^2 - 312500000t + 156250000.\end{aligned} \tag{21}$$

The solutions based on MHPM is found for the time steps $h = 10^{-2}$ and $h = 10^{-3}$ as tabulated in Table 2. The MHPM solutions at $h = 10^{-3}$ are of comparable accuracy with that of the Runge-Kutta-like method of [19] at the same step size.

4 Conclusions

In this paper, we presented the modified homotopy perturbation method (MHPM) for solving linearstiff IVPs. Direct applications of the classical HPM can fail for stiff problems. The MHPM is shown here to be a promising alternative method

Table 2. MHPM solutions using 6 terms as compared with the exact solutions and solutions from the explicit Runge-Kutta-like method [19] for example 2.

t	Exact, (20)	RK [19], $h = 10^{-3}$	MHPM $\phi_5, h = 10^{-2}$	MHPM, $\phi_6, h = 10^{-3}$
0.1	0.00835393217	0.00835393019	0.00846848100	0.00835393900
0.2	0.00684000045	0.00684000366	0.00693387200	0.00684000900
0.3	0.00560011874	0.00560012136	0.00567697500	0.00560012600
0.4	0.00458498943	0.00458499158	0.00464791500	0.00458499500
0.5	0.00375387185	0.00375387361	0.00380538800	0.00375387600
0.6	0.00307341033	0.00307341177	0.00311558970	0.00307341380
0.7	0.00251629555	0.00251629673	0.00255082750	0.00251629550
0.8	0.00206016855	0.00206016952	0.00208844460	0.00206017130
0.9	0.00168672335	0.00168672414	0.00170987040	0.00168672750
1.0	0.00138097228	0.00138097293	0.00139992420	0.00138097090

for stiff equations. In addition to the choice of time stepsize, the MHPM has the number of terms of the series solution as an extra parameter for controlling the accuracy of solutions. We note that the MHPM solutions were computed via a simple algorithm with less amount of computations and without any need for perturbation techniques, special transformations, linearization or discretization.

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