On Dynamical System Relating to Quantum Markov Chain Associated with Ising Model on Cayley Tree

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Abstract: In the present paper, we study stability of the dynamical system corresponding quantum Markov chain (QMC) associated with the Ising model on Cayley tree of order two. To study certain properties of QMC we reduce our investigation to the study of dynamics of a nonlinear dynamical system. For such a dynamical system it is proved existence of exactly three fixed points and absence of periodic points. Moreover, it is established finiteness and infiniteness of the trajectory of the system.

Key words: Cayley tree, quantum Markov chain, Ising model, stability of the dynamical system.

INTRODUCTION

It is know that Markov fields play an important role in classical probability, in physics, in biological and neurological models and in an increasing number of technological problems such as image recognition. Therefore, it is quite natural to forecast that the quantum analogue of these models will also play a relevant role. The quantum analogues of Markov processes were first constructed in (Accardi, L., 1975), where the notion of quantum Markov chain on infinite tensor product algebras was introduced. Nowadays, quantum Markov chains have become a standard computational tool in solid state physics, and several natural applications have emerged in quantum statistical mechanics and quantum information theory. The reader is referred to (Accardi, L., F. Fidaleo, 2003; Accardi, L., F. Fidaleo, 2003; Fidaleo, F., F. Mukhamedov, 2004; Fukui, Y., T. Horiguchi, 2000) and the references cited therein, for recent developments of the theory and the applications.

A first attempts to construct a quantum analogue of classical Markov fields has been done in (Accardi, L., F. Fidaleo, 2003; Accardi, L., F. Fidaleo, 2003; Accardi, L., V. Liebscher, 1999; Fidaleo, F., F. Mukhamedov, 2004). These papers extend to fields the notion of quantum Markov state introduced in (Accardi, L., A. Frigerio, 1983) as a sub–class of the quantum Markov chains introduced in (Accardi, L., 1975). Typically a system is identified to a point in a graph: if this graph is not isomorphic to an interval in (1–dimensional case). The crucial role of the localization is at the root of the difficulties to construct nontrivial examples of Markov fields. Gaussian states (quasi-free, in the physics terminology) also have a simple structure, but they do not describe physically interesting interactions. Note that in mentioned papers quantum Markov fields were considered over multidimensional integer lattice. This lattice has so called amenability condition. Therefore, it is natural to investigate quantum Markov fields over non-amenable lattices. One of the simplest non-amenable lattices is a Cayley tree. First attempts to investigate Quantum Makov chains over such trees was done in (Affleck, L., 1988), such studies were related to investigate thermodynamic limit of valence-bond-solid models on a Cayley tree (Fannes, M., 1992). The mentioned considerations naturally suggest the study of the following problem: the extension to fields the notion of generalized Markov chain. In the present paper using the construction of Quantum Markov Chains (QMC) defined on the Cayley tree of order two which is given in (Accardi, L.,) we obtain a nonlinear dynamical system related to Ising model. Our main goal is to study an asymptotical behavior of the obtained nonlinear dynamical system. To do this, we find all fixed point of the dynamical system and show an absence of periodic points. After this, we investigate the stability of the fixed points.

Fixed and Periodic Points of the Dynamical System:

Using the methods of (Accardi, L.,) we reduce the investigation of the quantum Markov chain asso ciated

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with the Ising model on the Cayley tree of order two to the following dynamical system $f : R_+^2 \to R_+^2$

$$\begin{cases} x' = \dfrac{2\theta\sqrt{x} - 2\sqrt{y}}{\theta^2 - 1} \\ y' = \dfrac{-2\sqrt{x} + 2\theta\sqrt{y}}{\theta^2 - 1} \end{cases} \tag{1}$$

where $\beta > 0$, $\quad \theta = \exp 2\beta$, and $(x', y') := f(x, y)$.

The following function $g_\theta : R \to R$

$$g_\theta(t) = \frac{\theta\sqrt{t} - 1}{\theta - \sqrt{t}} \tag{2}$$

helps to study asymptotical behavior of the dynamical system (1). One can see that the domain D of the function $g_\theta : R \to R$ is $[0, \theta^2) \cup (\theta^2, \infty)$.

One can easily show the following property of the function $g_\theta : D \to R$.

Proposition 1:

Let $g_\theta : D \to R$ be a function given by (2). Then the following assertions hold true:

- If $\theta > 3$ then g_θ has three fixed points, i.e.,

$$Fix(g_\theta) = \left\{ 1, t_\pm = \left(\frac{\theta - 1 \pm \sqrt{(\theta - 3)(\theta + 1)}}{2} \right)^2 \right\};$$

- If $1 < \theta \le 3$ then g_θ has one fixed point, i.e.,

$Fix(g_\theta) = \{1\}$;

- If $\theta > 3$ then $g_\theta(t) < t$ for any $t \in \left(\frac{1}{\theta^2}, t_- \right) \cup (1, t_+)$ and $g_\theta(t) > t$ for any $t \in (t_-, 1) \cup (t_+, \theta^2)$;

- If $1 < \theta \le 3$ then $g_\theta(t) < t$ for any $t \in \left(\frac{1}{\theta^2}, 1 \right)$ and $g_\theta(t) > t$ for any $t \in (1, \theta^2)$;

- g_θ is a strictly increasing function;

- g_θ does not have any k periodic points;

- g_θ is positive if and only if $t \in \left(\dfrac{1}{\theta^2}, \theta^2 \right)$.

Let us study an asymptotical behavior of the function $g_\theta : D \to R$.

Proposition 2:

Let $g_\theta : D \to R$ be a function given by (2) and $\theta > 3$. Then the following assertions hold true:

- If $t_0 \in (t_-, t_+)$ then the trajectory $\left\{ g_\theta^{(n)}(t_0) \right\}_{n=1}^\infty$, starting from the point t_0, converges to the fixed point which is equal to one;

- If $t_0 \in \left(\frac{1}{\theta^2}, t_- \right) \cup \left(t_+, \theta^2 \right)$ then the trajectory $\left\{ g_\theta^{(n)}(t_0) \right\}_{n=1}^\infty$, starting from the point t_0, is finite.

Sketch of the Proof:

Let us consider the case when $t_0 \in (t_-, t_+)$. Since the function g_θ is strictly increasing and $\frac{1}{\theta^2} < t_- < 1 < t_+ < \theta^2$, then the segment $[t_-, t_+]$ is an invariant with respect to g_θ and $g_\theta(t) > 0$ for any $t \in (t_-, t_+)$. Therefore, one can consider the trajectory $\left\{ g_\theta^{(n)}(t_0) \right\}_{n=1}^\infty$, starting from the point t_0 and it is infinite.

Without loss any generality, we may assume that $t_0 \in (t_-, 1)$. Then, according to Proposition 1, we have $g_\theta(t_0) > t_0$. Invariance of the segment $[t_-, 1]$ with respect to g_θ, yields monotonicity of the sequence

$$t_- < t_0 < g_\theta(t_0) < g_\theta^{(2)}(t_0) < \cdots < g_\theta^n(t_0) < \cdots < 1$$

Hence, $\left\{ g_\theta^{(n)}(t_0) \right\}_{n=1}^\infty$ is convergent, and continuity of $g_\theta[t_-, 1]$ implies that the limit of the sequence $\left\{ g_\theta^{(n)}(t_0) \right\}_{n=1}^\infty$ is a fixed point, which is equal to 1.

Analogously, one can show that if $t_0 \in (1, t_+)$ then the trajectory $\left\{ g_\theta^{(n)}(t_0) \right\}_{n=1}^\infty$, starting from the point t_0, is monotone decreasing and it converges to fixed point which is equal to one.

Let us consider the case when $t_0 \in \left(\frac{1}{\theta^2}, t_- \right) \cup \left(t_+, \theta^2 \right)$. Without loss any generality, we assume that $t_0 \in \left(\frac{1}{\theta^2}, t_- \right)$. We will show that the trajectory $\left\{ g_\theta^{(n)}(t_0) \right\}_{n=1}^\infty$, starting from the point t_0, is finite. Suppose the contrary, that is, the trajectory $\left\{ g_\theta^{(n)}(t_0) \right\}_{n=1}^\infty$, is infinite. Then according to Proposition 1 we have

$$g_\theta^{(n)}(t_0) > \frac{1}{\theta^2}, \tag{3}$$

for any $n \in N$. On the other hand, due to Proposition 1 we have the following monotone decreasing sequence

$$t_- > t_0 > g_\theta(t_0) > g_\theta^{(2)}(t_0) > \cdots > g_\theta^{(n)}(t_0) > \cdots \tag{4}$$

It follows from (3) and (4) that the sequence $\left\{ g_\theta^{(n)}(t_0) \right\}_{n=1}^\infty$ converges. Since g_θ is the continuous function on the segment $\left[\frac{1}{\theta^2}, t_- \right]$, then limit of the sequence $\left\{ g_\theta^{(n)}(t_0) \right\}_{n=1}^\infty$ should be fixed point which is less than t_-. But this is a contradiction, because the function g_θ does not have any fixed which is less than t_- (see Proposition 1). This completes the proof.

Analogously, one can prove the following proposition.

Proposition 3:

Let $g_\theta : D \to R$ be a function given by (2) and $1 < \theta \le 3$. For any initial point $t_0 \in \left(\dfrac{1}{\theta^2}, \theta^2 \right)$ the

trajectory $\left\{ g_\theta^{(n)}(t_0) \right\}_{n=1}^{\infty}$, starting from the point t_0, is finite.

By means Proposition 1, one can easily prove the following property of the dynamical system $f : R_+^2 \to R_+^2$ given by (1).

Theorem 1:

Let $f : R_+^2 \to R_+^2$ be a dynamical system be given by (1). Then the following assertions hold true:

• If $\theta > 3$ then f has three fixed points which are equal to $(A_k B_k, B_k^2)$, where

$$A_k = \frac{2\theta\sqrt{t_k} - 2}{\theta^2 - 1}, \ B_k = \frac{2\theta - 2\sqrt{t_k}}{\theta^2 - 1}, \ k = 1, 2, 3,$$

and $t_1 = 1$, $t_2 = t_-$, $t_3 = t_+$;

• If $1 < \theta \le 3$ then f has one fixed point which is equal to $(A_1 B_1, B_1^2)$.

• f does not have any k periodic points.

Stability of Fixed Points:

Let us study an asymptotical behavior of the dynamical system $f : R_+^2 \to R_+^2$ given by (1).

It is clear that the dynamical system (1) is well defined if and only if $x, y \ge 0$ and $\dfrac{1}{\theta^2} y \le x \le \theta^2 y$.

Proposition 4:

Let $f : R_+^2 \to R_+^2$ be a dynamical system be given by (1). Then the following assertions hold true:

• If $\theta > 3$ then f has three invariant lines l_k which are defined by $y = \dfrac{x}{t_k}$, $k = 1, 2, 3$, where $t_1 = 1$,

$t_2 = t_-$, $t_3 = t_+$;

• If $1 < \theta \le 3$ then f has one invariant line l_1 which is defined by $y = \dfrac{x}{t_1}$.

Sketch of the Proof:

It follows from (1) that if t_k is a fixed point of the function g_θ then $y = \dfrac{x}{t_k}$ yields $y' = \dfrac{x'}{t_k}$ Therefore,

if $\theta > 3$ then the dynamical system f has three invariant lines l_k which are defined by $y = \dfrac{x}{t_k}$, $k = 1, 2, 3$,

and if $1 < \theta \le 3$ then the dynamical system f has one invariant line l_1 which is defined by $y = \dfrac{x}{t_1}$.

Theorem 2:

Let $f : R_+^2 \to R_+^2$ be a dynamical system be given by (1) and $\theta > 3$. Then the following assertions hold true:

• If an initial point (x^0, y^0) belongs to an invariant line l_k (where $k = 1, 2, 3$) of the dynamical system

$f : R_+^2 \to R_+^2$ then the trajectory $\left\{ f^{(n)}(x^0, y^0) \right\}_{n=1}^{\infty}$, starting from the point (x^0, y^0), converges to a fixed

point $(A_k B_k, B_k^2)$ belonging to the line l_k;

- If an initial point (x^0, y^0) satisfies the following condition $\frac{x^0}{y^0} \in (t_-, 1) \cup (1, t_+)$ then the trajectory $\{f^{(n)}(x^0, y^0)\}_{n=1}^{\infty}$, starting from the point (x^0, y^0), converges to a fixed point $(A_1 B_1, B_1^2)$ belonging to l_1;

- If an initial point (x^0, y^0) satisfies the following condition $\frac{x^0}{y^0} \in \left(\frac{1}{\theta^2}, t_-\right) \cup (t_+, \theta^2)$ then the trajectory $\{f^{(n)}(x^0, y^0)\}_{n=1}^{\infty}$, starting from the point (x^0, y^0), is finite.

Sketch of the Proof:

Let us consider the case when an initial point (x^0, y^0) belongs to an invariant line l_k of the dynamical system $f : R_+^2 \to R_+^2$. We then have

$$\frac{x^{(n)}}{y^{(n)}} = t_k,$$

for any $n \in N$, where $(x^{(n)}, y^{(n)}) := f^{(n)}(x^0, y^0)$. In other words, one gets

$$\begin{cases} x^{(n+1)} = A_k \sqrt{y^{(n)}} \\ y^{(n+1)} = B_k \sqrt{y^{(n)}}. \end{cases}$$

One can see that

$$\begin{cases} x^{(n+1)} = A_k \underbrace{\sqrt{B_k \sqrt{B_k \cdots \sqrt{B_k}}}}_{n+1} \sqrt[2^{n+1}]{y^0} = A_k B_k^{\frac{\frac{1}{2} - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}}} \sqrt[2^{n+1}]{y^0} \\ \\ y^{(n+1)} = B_k \underbrace{\sqrt{B_k \sqrt{B_k \cdots \sqrt{B_k}}}}_{n+1} \sqrt[2^{n+1}]{y^0} = B_k^{\frac{1 - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}}} \sqrt[2^{n+1}]{y^0}. \end{cases}$$

Therefore, we obtain

$$\begin{cases} x^{(n+1)} = A_k B_k \sqrt[2^n]{y^0 B_k} \\ y^{(n+1)} = B_k^2 \sqrt[2^{n+1}]{y^0 B_k}. \end{cases}$$

Consequence, the sequence $\{(x^{(n)}, y^{(n)})\}_{n=0}^{\infty}$ converges to the fixed point $(A_k B_k, B_k^2)$ which belongs to the invariant line l_k.

Let us consider the case when an initial point (x^0, y^0) satisfies the following condition

$$\frac{x^0}{y^0} \in (t_-, 1) \cup (1, t_+).$$

It follows from (1) that

$$\frac{x^{(n+1)}}{y^{(n+1)}} = g\left(\frac{x^{(n)}}{y^{(n)}}\right).$$

According to Proposition 2, the sequence $\left\{ \dfrac{x^{(n)}}{y^{(n)}} \right\}_{n=0}^{\infty}$ converges to the fixed point $t_1 = 1$. Taking

$$g\left(\frac{x^{(n)}}{y^{(n)}} \right) = a_n, \; b_n = \frac{2\theta - 2\sqrt{a_n}}{\theta^2 - 1}, \; c_n = \frac{2\theta\sqrt{a_n} - 2}{\theta^2 - 1},$$

we then have

$$\begin{cases} x^{(n+1)} = c_n\sqrt{y^{(n)}} \\ y^{(n+1)} = b_n\sqrt{y^{(n)}}. \end{cases}$$

This means

$$\begin{cases} x^{(n+1)} = c_n \underbrace{\sqrt{b_{n-1}\sqrt{b_{n-2}\cdots\sqrt{b_0}}}}_{n+1} \, \sqrt[2^{n+1}]{y^0} \\ y^{(n+1)} = b_n \underbrace{\sqrt{b_{n-1}\sqrt{b_{n-2}\cdots\sqrt{b_0}}}}_{n+1} \, \sqrt[2^{n+1}]{y^0}. \end{cases}$$

The following Lemma is useful for calculating the limit point of the sequence $\left\{ \left(x^{(n)}, y^{(n)} \right) \right\}_{n=0}^{\infty}$.

Lemma:

If $b_n > 0$ for any $n \in N$ and the sequence $\{b_n\}_{n=0}^{\infty}$ converges to $b > 0$ then the sequence

$$d_n = b_n \underbrace{\sqrt{b_{n-1}\sqrt{b_{n-2}\cdots\sqrt{b_0}}}}_{n+1}$$

converges to b^2.

We know that

$$a_n \to 1, \; b_n \to B_1, \; c_n \to A_1.$$

Then, according to Lemma, the sequence $\left\{ \left(x^{(n)}, y^{(n)} \right) \right\}_{n=0}^{\infty}$ converges to $\left(A_1 B_1, B_1^2 \right)$ which belongs to an invariant line l_1.

Let us consider the case when an initial point $\left(x^0, y^0 \right)$ satisfies the following condition

$$\frac{x^0}{y^0} \in \left(\frac{1}{\theta^2}, t_- \right) \cup \left(t_+, \theta^2 \right).$$

It follows from (1) that

$$\frac{x^{(n+1)}}{y^{(n+1)}} = g\left(\frac{x^{(n)}}{y^{(n)}} \right).$$

According to Proposition 2, the sequence $\left\{\dfrac{x^{(n)}}{y^{(n)}}\right\}_{n=0}^{\infty}$ is finite. Therefore, the sequence $\left\{\left(x^{(n)}, y^{(n)}\right)\right\}_{n=0}^{\infty}$ should be finite. This completes the proof.

By means Proposition 3, analogously, one can prove the following result.

Theorem 3:

Let $f: R_+^2 \to R_+^2$ be a dynamical system be given by (1) and $1 < \theta \le 3$. Then the following assertions hold true:

- If an initial point $\left(x^0, y^0\right)$ belongs to an invariant line l_1 of the dynamical system $f: R_+^2 \to R_+^2$ then the trajectory $\left\{f^{(n)}\left(x^0, y^0\right)\right\}_{n=1}^{\infty}$, starting from the point $\left(x^0, y^0\right)$, converges to a fixed point $\left(A_1 B_1, B_1^2\right)$ which belongs to an invariant line l_1;

- If an initial point $\left(x^0, y^0\right)$ satisfies the following condition $\dfrac{x^0}{y^0} \in \left(\dfrac{1}{\theta^2}, 1\right) \cup \left(1, \theta^2\right)$ then the trajectory $\left\{f^{(n)}\left(x^0, y^0\right)\right\}_{n=1}^{\infty}$, starting from the point $\left(x^0, y^0\right)$, is finite.

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