

HOMOTOPY OF VOLTERRA QUADRATIC STOCHASTIC OPERATORS

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Let

$$S^{m-1} = \left\{ x = (x_1, x_2, \dots, x_m) \in \mathbb{R}^m : \sum_{k=1}^m x_k = 1, x_k \geq 0 \right\}$$

be $(m-1)$ -dimensional simplex.

A Volterra quadratic stochastic operator (q.s.o.) $V : S^{m-1} \rightarrow S^{m-1}$ is defined by the formula

$$(V(x))_k = x_k \left(1 + \sum_{i=1}^m a_{ki} x_i \right), \quad k = \overline{1, m},$$

where $a_{ki} = -a_{ik}$, $|a_{ki}| \leq 1$, i.e. $A_m = (a_{ki})_{k,i=1}^m$ is a skew-symmetrical matrix.

Definition. A skew-symmetrical matrix A_m is called transversal if all even order main minors are nonzero. We say that a Volterra q.s.o. V is transversal if the corresponding skew-symmetrical matrix A_m is transversal. Let $\text{Fix}(V) = \{x \in S^{m-1} : Vx = x\}$ be the set of all fixed points of the Volterra q.s.o. Note that the set $\text{Fix}(V)$ of any (transversal) Volterra q.s.o. is finite.

Definition. Two transversal Volterra q.s.o. V_0, V_1 are called homotopic, if there exists a family of transversal Volterra operators $\{V_\lambda\}_{\lambda \in [0,1]}$ continuously depending on λ , $V_\lambda|_{\lambda=0} = V_0$, $V_\lambda|_{\lambda=1} = V_1$ and $|\text{Fix}(V_\lambda)| = |\text{Fix}(V_0)| = |\text{Fix}(V_1)|$ for any $\lambda \in [0, 1]$.

The set of all Volterra q.s.o. forms a $\left(\frac{m(m-1)}{2}\right)$ -dimensional cube \mathcal{V}^{m-1} in $\mathbb{R}^{\frac{m(m-1)}{2}}$. Now let us consider the following set

$$\mathcal{M} = \left\{ V \in \mathcal{V}^{m-1} \mid gp_{i_1 i_2 \dots i_{2k}}(V) = 0, \exists i_1, i_2, \dots, i_{2k} \in I = \{1, 2, \dots, m\} \right\};$$

where $gp_{i_1 i_2 \dots i_{2k}}(V)$ is a even order main subpfaffian of the skew-symmetrical matrix A_m (see [1]). This set divides the cube \mathcal{V}^{m-1} into several connected components.

Theorem. Two transversal Volterra q.s.o. V_0, V_1 are homotopic if and only if the operators V_0 and V_1 belong to a connected component of $\mathcal{V}^{m-1} \setminus \mathcal{M}$.

Reference

1. Proskuryakov I. V. Problems in linear algebra, Moscow: Mir Publishers, 1978.