In: Advances in Mathematics Research. Volume 12 ISBN: 978-1-61761-899-4
Editors: Albert R. Baswell

Chapter 3

A LAPLACE'S PRINCIPLE BASED APPROACH FOR SOLVING FUZZY MATRIX GAMES

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ABSTRACT

We introduce a solution for matrix games with fuzzy payoffs via the α-cuts and the introduction of Nature as a third player expressing the uncertainty involved in the game. The beliefs of players about the behavior of Nature are based on the Laplace's principle of "insufficient reason". Moreover, we provide a procedure for computing the introduced solution.

Keywords: Matrix game, fuzzy payoff, Nash equilibrium, Laplace's principle.

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1. INTRODUCTION

Several approaches for solving matrix games with fuzzy payoffs have been proposed in literature (Bector et al., 2004a; Bector et al., 2004b; Bector & Chandra, 2005; Campos,1989; Li,1999, Li & Yang, 2004; Maeda, 2003; Nishizaki & Sakawa, 1995, 1997, 2001; Vijay et al, 2005a, 2005b, Vijay et al., 2007). These approaches can be classified into three classes. In the first class (Bector et al, 2004a; Bector et al., 2004b; Bector & Chandra, 2005; Campos, 1989; Li, 1999; Maeda; 2003; Vijay et al., 2005a, 2005b), following the theory of crisp matrix games, the problem of finding a Nash equilibrium Nash (1951), is transformed into a problem of resolution of a pair of independent fuzzy linear (single objective or multiobjective) programming problems. Then some defuzzification technique is used to transform the obtained pair of fuzzy linear programming problems into a pair of crisp linear programming problems. In Bector et al (2004b) the defuzzification of the pair of fuzzy linear programming problems is based on ranking of fuzzy numbers by Yager's different ranking functions (indexes) Yager (1981). In Bector et al. (2004a) a pair of dual fuzzy linear programming problems in the fuzzy sense is obtained, then the Yager's first index is used for defuzzification. In Li (1999) an ordering of triangular fuzzy numbers is used to define a solution to the fuzzy matrix game. Then based on the same ordering, a pair of two multiobjective crisp linear programming problems is obtained for finding the introduced solution. In Maeda (2003) a more general ordering of fuzzy numbers based on α-cuts is used. The introduced concept of solution is obtained as Nash equilibrium of a constructed crisp bimatrix game. It has to be noted that the approach in Maeda (2003) is limited to symmetric triangular fuzzy numbers only. Most of the approaches of the first class are discussed in details in Bector and Chandra (2005). In the second class Nishizaki & Sakawa (1995, 1997, 2001), the membership function of the expected fuzzy score $x^T \tilde{A} y$ of the maximizing player is used to define solutions. In Nishizaki & Sakawa (1995) the expected fuzzy score and a fuzzy goal are used to define a crisp payoff for the maximizing player as a degree of attainment of his fuzzy goal. The considered solution is based on the maxmin principle. Recently, Vijay et al. (2007) introduced a third class of approaches based on fuzzy relations. The fuzzy relation approach unifies the existing theories on fuzzy matrix games. In this approach the fuzzy matrix game is transformed into a pair of fuzzy optimization problems where constraints are expressed by fuzzy relations. Further, this pair is formulated as a pair of semi-infinite programming problems. Matrix games with fuzzy goals and matrix games with possibility and necessity relations have been studied as special cases. Finally, note that most of the approaches developed for matrix games with fuzzy payoffs have been extended to bimatrix games with fuzzy payoffs and fuzzy goals, and multiple objective matrix and bimatrix games with fuzzy payoffs and fuzzy goals (Bector & Chandra, 2005; Nishizaki & Sakawa, 2001; and Vijay et al., 2005a, 2005b). For an extensive survey on fuzzy bimatrix games and fuzzy games in normal form and their applications see Larbani (2009a).

In Chen and Larbani (2005) we proposed a game approach for solving a multi-attribute decision making (MADM) problem with fuzzy decision matrix by using α-cuts and introducing Nature as a player (against the decision maker) representing the fuzzyness. In Larbani (2009b) we have extended this approach to bimatrix games with fuzzy payoffs by using α-cuts and introducing Nature as a third player representing the uncertainty involved in such games. The proposed solution of the bimatrix game is based on the maxmin principle of decision making under uncertainty Luce and Raiffa (1957).

In this paper we adopt the same approach for solving matrix games with fuzzy payoffs, however, the solution we propose here is based on the Laplace's principle of "insufficient reason" for decision making under uncertainty Luce and Raiffa (1957). We first use α-cuts to defuzzify the payoff matrix, then we construct a crisp three person game where Nature is introduced as a third player without payoff function that chooses its strategies from the α-cuts of the entries of the payoff matrix. Then using the principle of "insufficient reason" of Laplace for decision making under uncertainty Luce & Raiffa (1957), we introduce a solution concept to the defuzzified fuzzy matrix game. The computation of our solution is also studied. A discussion of the existing solutions of fuzzy bimatrix games is provided.

The rest of the paper is organized as follows. Section 2 presents the proposed solution. Section 3 deals with the computation of the introduced solution and discussion of related work. Section 4 concludes the paper.

2. The Solution

In this section we present a solution concept for a matrix games with fuzzy payoffs based on defuzzification of the fuzzy payoff matrix via α-cuts and the introduction of a third player without payoff function, Nature. Here

Nature represents the uncertainty involved in the game i.e. fuziness. We assume that the players adopt the Laplace's principle of "insufficient reason" for decision making under uncertainty Luce & Raiffa (1957) with respect to the behavior of Nature towards them.

Let us consider the following matrix game with fuzzy payoffs

$$G_1 = (S^m, S^{\mathrm{n}}, \tilde{A}) \tag{1}$$

where $S^m = \{\, x = (x_1, x_2, ..., x_m)\,,\ x_i \geq 0\,,\ i = \overline{1,m}\,,\ \sum_{i=1}^{m} x_i = 1\}$,

$S^n = \{\, y = (y_1, y_2, ..., y_n)\,,\ y_i \geq 0\,,\ i = \overline{1,n}\,,\ \sum_{j=1}^{n} y_i = 1\}$ are the sets of mixed strategies of Player I and II respectively, $\tilde{A} = (\tilde{a}_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$ is the fuzzy payoff matrix, $\tilde{a}_{ij}$ is a fuzzy interval with bounded support as defined by Dubois & Prade (2000). A fuzzy interval $\tilde{F}$ with bounded support is defined by $\tilde{F} = (R, \mu(.))$ with $\mu_{\tilde{F}}(.) : R \to [0,1]$ verifying the following conditions

(i) $\mu_{\tilde{F}}(x) = 0$ for all $x \in]-\infty, c]$,

(ii) $\mu_{\tilde{F}}(.)$ is right-continuous non-decreasing on $[c,a]$,

(iii) $\mu_{\tilde{F}}(x) = 1$ for all $x \in [a,b]$,

(iv) $\mu_{\tilde{F}}(.)$ is left-continuous non-increasing on $[b,d]$,

(v) $\mu_{\tilde{F}}(x) = 0$ for all $x \in [d, +\infty[$,

where $-\infty < c \leq a \leq b \leq d < +\infty$, and R is the real line.

Assumption 2.1. In non cooperative games, the following assumptions are generally made.

(i) Players are rational.
(ii) There are no enforceable agreements between players.

(iii) The game is of complete information, that is, all the data of the game (1) are common knowledge among players Fudenberg & Tirole (1993).

Regarding the fuzzy payoffs, it is assumed that the only information available to players about the fuzzy payoff $\tilde{a}_{ij}$ is its membership function $\mu_{\tilde{a}_{ij}}(x)$, $i=\overline{1,m}$, $j=\overline{1,n}$.

We start by taking the α-cut $\tilde{A}^{\alpha}=([\tilde{a}_{ij}]^{\alpha})_{1\le i\le m, 1\le j\le n}$ of the payoff matrix $\tilde{A}$ of the game (1), where $[\tilde{a}_{ij}]^{\alpha}=\{a_{ij} / \mu_{\tilde{a}_{ij}}(a_{ij})\ge\alpha\}$, $i=\overline{1,m}$, $j=\overline{1,n}$. Since $\tilde{a}_{ij}$ is a fuzzy interval with bounded support, its α-cut is a closed bounded interval, that is

$$[\tilde{a}_{ij}]^{\alpha}=\{a_{ij} / \mu_{\tilde{a}_{ij}}(a_{ij})\ge\alpha_{ij}\}=[a_{ij}^{L\alpha}, a_{ij}^{U\alpha}],\ i=\overline{1,m},\ j=\overline{1,n}.$$

We can also write

$$[\tilde{a}_{ij}]^{\alpha}=[a_{ij}^{L\alpha}, a_{ij}^{U\alpha}]=\left\{\beta_{ij}a_{ij}^{U\alpha}+(1-\beta_{ij})_{ij}^{L\alpha} / \beta_{ij}\in[0,1]\right\},\ i=\overline{1,m},\ j=\overline{1,n}. \quad (2)$$

Thus, choosing a number a_{ij} in the α-cut $[\tilde{a}_{ij}]^{\alpha}$ is equivalent to choosing a number β_{ij} in [0, 1], $i=\overline{1,m}$, $j=\overline{1,n}$. The α-cut $[\tilde{a}_{ij}]^{\alpha}$ is the set of payoffs a_{ij} that have at least an α degree of membership to $\tilde{a}_{ij}$.

In this paper we assume that once the cut-level α has been chosen, the players are certain that their payoff value a_{ij} will vary in $[\tilde{a}_{ij}]^{\alpha}$, $i=\overline{1,m}$, $j=\overline{1,n}$. However, we assume that they do not know which particular values $a_{ij}\in[\tilde{a}_{ij}]^{\alpha}$ (or equivalently β_{ij} in [0, 1]), $i=\overline{1,m}$, $j=\overline{1,n}$ will actually occur. In terms of necessity measure Dubois and Prade (2000), we assume that once α is chosen, the players assign a necessity degree 1 to the event that the payoff value a_{ij} is in $[\tilde{a}_{ij}]^{\alpha}$, $i=\overline{1,m}$, $j=\overline{1,n}$. Hence, the players would consider that a_{ij} is an unknown parameter that varies in $[\tilde{a}_{ij}]^{\alpha}$, $i=\overline{1,m}$,

$j=\overline{1,n}$. Thus, in addition to the strategic uncertainty, the players face another type of uncertainty represented by the possible realizations of the unknown parameters (payoffs) a_{ij} in $[\tilde{a}_{ij}]^{\alpha}$, $i=\overline{1,m}$, $j=\overline{1,n}$. Hence, they have to adopt a decision making under uncertainty principle Luce & Raiffa (1957). Then the problem can be considered as a game against Nature Milnor (1957). Thus, Nature enters the game as a third player that chooses the crisp payoffs a_{ij} in $[\tilde{a}_{ij}]^{\alpha}$, $i=\overline{1,m}$, $j=\overline{1,n}$. However, Nature is a special player: it has no payoff function. Here it has to be noted that since the interests of players are totally opposed (zero-sum game), Nature cannot be against both of them at the same time: if Nature is against any player, this means that it favors the other player. Thus, we have two possible approaches. The first one is to assume that Nature is neutral or has a balanced behavior towards players. The second one is that Nature is against one of the players, in which case it favors the other player. To illustrate this fact let us consider any strategy β_{ij} of Nature as defined in (2). Then if β_{ij} is greater than 1/2, this means that Nature favors larger values of a_{ij}, that is, it favors Player I and goes against Player II, with respect to strategies *i* and *j*. Conversely, if it chooses β_{ij} less than 1/2, then it favors Player II and goes against player I. Finally, if Nature chooses β_{ij} = ½, then it shows a balanced behavior towards players with respect to strategies *i* and *j*. Globally, the behavior of Nature towards players can be described as follows. According to (2), choosing a_{ij} in $[\tilde{a}_{ij}]^{\alpha}$, $i=\overline{1,m}$, $j=\overline{1,n}$ is equivalent to choosing β_{ij} in [0, 1], $i=\overline{1,m}$, $j=\overline{1,n}$. The set of all possible choices (or behaviors) of Nature is

$$T=\left\{\beta=(\beta_{ij})\,/\,\beta_{ij}\in[0,1],\ i=\overline{1,m},\ j=\overline{1,n}\right\}.$$

First, notice that if Nature chooses a strategy $\beta=(\beta_{ij})$ in T such that $\beta_{ij}=1$, for all $i=\overline{1,m}$, $j=\overline{1,n}$, i.e. $\sum_{i,j}\beta_{ij}=mn$ (here and in the sequel we denote by $\sum_{i,j}$ the double summation symbol $\sum_{i=1}^{m}\sum_{j=1}^{n}$), this means that it favors the Player I completely. On the other hand, if Nature chooses a strategy $\beta = (\beta_{ij})$ in T such that $\beta_{ij} = 0$, for all $i = \overline{1,m}$, $j = \overline{1,n}$, i.e. $\sum_{i,j} \beta_{ij} = 0$, this means that Nature favors Player II completely. For any strategy $\beta = (\beta_{ij})$ in T of Nature, we have $0 \le \sum_{i,j} \beta_{ij} \le mn$, where the values 0 and *mn* represent the extreme behaviors. Any other type of behavior of Nature can be represented by a value $p \in [0,1]$, such that *p* is the degree to which Nature favors Player I and 1-*p* is the degree to which Nature favors Player II. Indeed, for a given $p \in [0,1]$, the set of strategies of Nature that corresponds to this behavior is

$$T_p = \left\{ \beta = (\beta_{ij}) / \beta \in T \text{ and} \sum_{i,j} \beta_{\text{ij}} = pmn \right\} \tag{3}$$

Thus, we obtain the following three-person extended crisp game

$$G_2 = (S^m, S^n, T_p, A_\beta) \tag{4}$$

In this game there are three players: Player I, Player II and Nature. Nature has no payoff, its set of strategies is T_p; the payoff functions and sets of strategies of Player I and Player II are $x^T A_\beta y$, S^m and $-x^T A_\beta y$, S^n respectively; $A_\beta = (\beta_{ij}(a_{ij}^{U\alpha} - a_{ij}^{L\alpha}) + a_{ij}^{L\alpha})_{1\le i\le m, 1\le j\le n}$ is the crisp payoff matrix of the game.

The idea of introducing variables that take values in α-cuts to defuzzify a decision making problem was first introduced by Sakawa & Yano (1989) for a multiobjective problem with fuzzy parameters, but they assumed that the introduced variables that vary in the α-cuts are decision variables. That is, they are controlled by the decision makers. Our approach differs in the sense that we assume that the decision makers control the level α of the α-cuts, but they do not control the variables a_{ij} that take their values in the α-cuts $[\tilde{a}_{ij}]^\alpha$, $i = \overline{1,m}$, $j = \overline{1,n}$. In Chen and Larbani (2005) we have first used this approach to solve a multi-attribute decision making (MADM) problem with fuzzy decision matrix by introducing Nature as a second player. We made the assumption that the entries of the decision matrix are interdependent triangular fuzzy numbers via a parameter λ. In Larbani (2009b) we have extended this approach to bimatrix games with fuzzy payoffs by using α-cuts and introducing Nature as a third player representing the fuzziness involved in such game. The proposed solution of the bimatrix game is based on the maxmin principle of decision making under uncertainty Luce and Raiffa (1957). In this paper we adopt the same approach for solving fuzzy matrix games with fuzzy payoffs, however, the solution we propose here is based on the Laplace's principle of "insufficient reason" for decision making under uncertainty Luce and Raiffa (1957).

Definition 2.1. Assume that $p \in [0,1]$ and the cut level α are given. A pair of real numbers (v, w) is called αp- acceptable solution to the game (4) if there exists a pair of mixed strategies $(x^*, y^*) \in S^m \times S^n$ such that

(i) $x^{*T} A_\beta y \geq v$, $\forall y \in S^n$ and $\forall \beta \in T_p$,

(ii) $x^T A_\beta y^* \leq w$, $\forall x \in S^m$ and $\forall \beta \in T_p$.

If (v, w) is an αp-acceptable solution to the game (4) then v (respectively w) is called an αp-acceptable value for Player I (respectively for Player II).

Let us introduce the following sets for Player I and Player II respectively

$$W_1^{\alpha p} = \{v \in R / \exists x' \in S^m,\ x'^t A_\beta y \geq v,\ \forall (y, \beta) \in S^n \times T_p\} \text{ and}$$

$$W_2^{\alpha p} = \{w \in R / \exists y' \in S^n, x^t A_\beta y' \leq w,\ \forall (x, \beta) \in S^m \times T_p\}.$$

The relation between the αp-acceptable solution and the sets $W_1^{\alpha p}$ and $W_2^{\alpha p}$ is given by the following proposition the proof of which is straightforward.

Proposition 2.1. Assume that $p \in [0,1]$ and the cut level α are given. A pair of real numbers (v^*, w^*) is an $\alpha\, p$-acceptable solution for the game (4) if and only if $(v^*, w^*) \in W_1^{\alpha p} \times W_2^{\alpha p}$.

Proposition 2.2. $MaxW_1^{\alpha p}$ and $MinW_2^{\alpha p}$ exist.

Proof. The function $f(x, y, \beta) = x^t A_\beta y$ is continuous in the compact $S^m \times S^n \times T_p$, then the value $\max_x \min_{(\beta,y)} x^t A_\beta y = \varepsilon$ exists. Therefore, there exists $x' \in S^m$ such that $x'^t A_\beta y \geq \varepsilon, \forall (y, \beta) \in S^n \times T_p$, hence $\varepsilon \in W_1^{\alpha p}$, that is, $W_1^{\alpha p}$ is not empty. The non emptiness of $W_2^{\alpha p}$ can be proved similarly. For a pair (y, β), we consider the function

$$f_{\beta y} : S^m \to IR$$

$$x \to f_{\beta,y}(x) = x^t A_\beta y - v$$

We have $v^* \in W_1^{\alpha p} \Leftrightarrow \exists x' \in S^m, f_{\beta,y}(x') \geq 0, \ \forall (y, \beta) \in S^n \times T_p$.

Then $W_1^{\alpha p} = \bigcap_{(\beta,y)\in S^n\times T} f_{\beta,y}{}^{-1}([0,+\infty[)$. The function $x \to f_{\beta,y}(x)$ is continuous, $\forall (y, \beta) \in S^n \times T_p$ and $[0,+\infty[$ is a closed set, then $f_{\beta,y}{}^{-1}([0,+\infty[)$, $\forall (y, \beta) \in S^n \times T_p$. Therefore $W_1^{\alpha p} = \bigcap_{(\beta,y)\in S^n\times T} f_{\beta,y}{}^{-1}([0,+\infty[)$ is a closed set. Let $v \in W_1^{\alpha p}$ then there exists $x' \in S^m$ such that $\forall (y, \beta) \in S^n \times T_p \ x'^t A_\beta y \geq v,$, which implies $\min_{(\beta,y)} x'^t A_\beta y \geq v$. Moreover, we have $\varepsilon = \max_x \min_{(\beta,y)} x^t A_\beta y \geq \min_{(\beta,y)} x'^t A_\beta y \geq v$. Since v is arbitrarily chosen in $W_1^{\alpha p}$, we conclude that $W_1^{\alpha p}$ is bounded from above by ε. Therefore, $MaxW_1^{\alpha p}$ exists. One can similarly show that $MinW_2^{\alpha p}$ exists.

Thus, we propose the following solution for the game (1).

Definition 2.2. Let $W^{\alpha p}{}_1$ and $W^{\alpha p}{}_2$ be the sets of $\alpha\, p$-acceptable values for Player I and player II, respectively. Let $(v^*, w^*) \in W^{\alpha p}{}_1 \times W^{\alpha p}{}_2$ such that $v^* = Max\, W^{\alpha p}{}_1$ and $w^* = Min\, W^{\alpha p}{}_2$ and let $(x^*, y^*) \in S^m \times S^n$ be the corresponding pair of mixed strategies. Then (x^*, y^*, v^*, w^*) is called $\alpha\, p$-Nash solution of the game (1), where v^* (respectively w^*) is the value of the game for player I (respectively Player II) and x^* (respectively y^*) is called an optimal strategy for Player I (respectively Player II).

The solution defined through Definitions 2.1-2.2 is similar to that of Bector et al. (2004a). However, our solution depends on the strategy of Nature β. Moreover, the inequalities (i)-(ii) in Definition 2.1 are nonlinear and crisp, which is not the case in Bector et al. (2004a).

According to the settings of the game, there is no reason to think that Nature will favor any of the players globally. Hence we can fairly assume that the players adopt the Laplace's insufficient reason principle of decision making under uncertainty Luce & Raiffa (1957) with respect to the behavior of Nature towards them. That is, the players assume that, globally, Nature has a balanced behavior towards them, but each of them does not know in which payoffs (entries a_{ij}) Nature favors him and in which payoff it does not. In order to maintain the complete information spirit of the game (1) (Assumption 2.1 (iii)), we assume that this assumption is a common knowledge among players as well. Let us represent formally this assumption on the behavior of Nature. In terms of strategies, this assumption means that Nature chooses strategies $\beta = (\beta_{ij})$ in T that yield a global value $\sum_{i,j} \beta_{ij}$ in the middle of the interval [0, *mn*] i.e.

$$\sum_{i,j} \beta_{ij} = \frac{0 + mn}{2} = \frac{mn}{2}$$

In other words, the players assume $p = 1/2$ in (3), i.e. Nature chooses its strategies β in

$$T_{1/2} = \left\{ \beta = (\beta_{ij}) \,/\, \beta \in T \text{ and } \sum_{i,j} \beta_{\text{ij}} = (1/2)mn \right\} \tag{5}$$

This means that, globally, Nature does not favor any of the players. The ideal case is when Nature chooses $\beta_{ij} = 1/2$, $i = \overline{1,m}$, $j = \overline{1,n}$, then it doesn't favor any of the two player globally or locally.

Definition 2.3. In the particular case where p = 1/2, an α 1/2-Nash solution of the game (1) is called α-Nash-Laplace solution (α-NL solution) of the game (1).

3. COMPUTATION OF THE SOLUTION

We will present a general method for computation of α-NL solution of the game (1).

Using Definitions 2.1-2.3, we can find an α-NL solution of the game (1) by solving the following pair of crisp nonlinear programming problems

Max v

Subject to,

$$x^T A_\beta y \geq v, \; y \in S^n \text{ and } \beta \in T_p \tag{6}$$

$$x \in S^m$$

and

$$Min \; w$$

Subject to,

$$x^T A_\beta y \leq w, \; x \in S^m \text{ and } \beta \in T_p \tag{7}$$

$y \in S^n$.

Since S^m and S^n are polytopes, we can use their extreme points only Vijay et al. (2005b), then the problems (6)-(7) can be transformed into a pair of crisp nonlinear programming problems

Max v

Subject to,

$$x^T (A_\beta)_j \geq v \;,\; j=\overline{1,n}\,,\; \beta \in T_p \,, \tag{8}$$

$$x \in S^m$$

and

Min w

Subject to,

$$(A_\beta)_i\, y \leq w \;,\; i=\overline{1,m}\,,\; \beta \in T_p \tag{9}$$

$$y \in S^n \,,$$

where $(A_\beta)_i$ (respectively $(A_\beta)_j$) denotes the *i*-th row (respectively *j*-th column) of the matrix A_β. Here also we note that the problems (8)-(9) are similar to the problems obtained in (Bector et al., 2004a; Campos, 1989). However, the problems (8)-(9) involve the additional variable β, which is not the case in (Bector et al., 2004a; Bector et al., 2004b).

Remark 3.1. Since the sets T_p, S^m and S^n are compact and the function $(x, y, \beta) \to x^T A_\beta y$ is continuous, then each of the problems (8)-(9) has a solution. Hence the problems (6)-(7) have a solution as well.

Remark 3.2.

(i) In the case the entries $\tilde{a}_{ij}$ of the fuzzy payoff matrix $\tilde{A}$ of the game (1) are symmetric triangular fuzzy numbers (STFNs), i.e. $\tilde{a}_{ij}$ = (a_{ij}^M ,h_{ij}), where a_{ij}^M is the main value and h_{ij} the width, we have $(a_{ij}^{U\alpha} - a_{ij}^{L\alpha}) = h_{ij}(2-2\alpha)$, hence $A_\beta = (\beta_{ij}(a_{ij}^{U\alpha} - a_{ij}^{L\alpha}) + a_{ij}^{L\alpha})_{1\leq i\leq m, 1\leq j\leq n} =$

$(\beta_{ij}\ h_{ij}(2-2\alpha)+a_{ij}^{L\alpha})_{1\le i\le m,1\le j\le n}$, then the matrix A_β takes the form $A_\beta = (2-2\alpha)\beta_h + A^{L\alpha}$, where $A^{L\alpha} = (a_{ij}^{L\alpha})_{1\le i\le m,1\le j\le n}$ and $\beta_h = (h_{ij}\beta_{ij})_{1\le i\le m,1\le j\le n}$.

(ii) In the case the entries $\tilde{a}_{ij}$ of the fuzzy payoff matrix $\tilde{A}$ of the game (1) are triangular fuzzy numbers (TFNs) Bector & Chandra (2005), i.e. $\tilde{a}_{ij}$ =($a_{ij}^L, a_{ij}^M, a_{ij}^U$), we have

$$[\tilde{a}_{ij}]^\alpha = [(a_{ij}^M - a_{ij}^{L\alpha})\alpha + a_{ij}^{L\alpha}, -(a_{ij}^{U\alpha} - a_{ij}^M)\alpha + a_{ij}^{U\alpha}].$$

Procedure 3.1. Computation of α p-Nash solution.

Step 1. Assume that p is fixed. Ask the players to provide their α-cut levels. Assume that they have chosen α_1 and α_2, respectively. Then take α =*Max*{ α_1, α_2 } in order to satisfy each player's choice. Further, compute the α-cuts

$$[\tilde{a}_{ij}]^\alpha = \{ a_{ij} / \mu_{\tilde{a}_{ij}}(a_{ij}) \ge \alpha \} = [a_{ij}^{L\alpha}, a_{ij}^{U\alpha}],\ i = \overline{1,m},\ j = \overline{1,n}$$

In case the players are not able to provide their own α-cut levels, they may be determined by consulting with experts.

Step 2. Construct and solve the pair of nonlinear programming problems (8)-(9). The obtained solution (x^*, y^*, v^*, w^*) is an α-NL solution to the considered game with cut-level α.

Let us now illustrate this procedure by an example. We consider the following example that appears in (Bector et al., 2004a; Bector & Chandra,2005 (see page 148); Campos, 1989; Li & Yang, 2004; Vijay et al., 2007), in order to compare our results with those of existing approaches.

Example 3.1. (Computation of α-NL solution) Consider the fuzzy game defined by the fuzzy payoff matrix

$$\tilde{A} = \begin{pmatrix} 1\tilde{8}0 & 1\tilde{5}6 \\ 9\tilde{0} & 1\tilde{8}0 \end{pmatrix} \tag{10}$$

The entries of $\tilde{A}$ are the following TFNs: $1\tilde{8}0 = (175,180,190)$, $1\tilde{5}6 = (150,156,158)$ and $9\tilde{0} = (80,90,100)$. Let us compute an α-NL solution of this game by Procedure 3.1, assuming p = 1/2.

Step1. Ask the players to provide their α-cut levels. Assume that they have chosen α_1 and α_2 respectively. Let α =Max{ α_1, α_2 } in order to satisfy both players.

First we solve the problem for arbitrary levels α_1, α_2, then we assume concrete values for these parameters. We have

$$[\tilde{a}_{ij}]^{\alpha} = \{ a_{ij} / \mu_{\tilde{a}_{ij}}(a_{ij}) \geq \alpha \} = [a_{ij}^{L\alpha}, a_{ij}^{U\alpha}],\ i = \overline{1,2},\ j = \overline{1,2}$$

$$[1\tilde{8}0]^{\alpha} = [(180-175)\alpha + 175,\ -(190-180)\alpha + 190]$$

$$= [5\alpha + 175,\ -10\alpha + 190],$$

$[156]^{\alpha} = [150 + 6\alpha, 158 - 2\alpha]$, and $[90]^{\alpha} = [80 + 10\alpha, 100 - 10\alpha]$.

We have $mnp = 2 \times 2 \times 1/2$ =2, then

$$T_{1/2} = \left\{ \beta = (\beta_{ij}) / \beta \in T \text{ and } \sum_{i,j} \beta_{ij} = 2 \right\}.$$

Compute the matrix $A_{\beta} = (\beta_{ij}(a_{ij}^{U\alpha} - a_{ij}^{L\alpha}) + a_{ij}^{L\alpha})_{1 \leq i \leq m, 1 \leq j \leq n}$, $\beta \in T_{1/2}$. Since the payoffs are TFN, according to Remark 3.2

$$A_{\beta} = (\beta_{ij}(a_{ij}^{U\alpha} - a_{ij}^{L\alpha}) + a_{ij}^{L\alpha})_{1 \leq i \leq m, 1 \leq j \leq n} =$$

$$\begin{pmatrix} (15 - 15\alpha)\beta_{11} + (175 + 5\alpha) & (8 - 8\alpha)\beta_{12} + (150 + 6\alpha) \\ (20 - 20\alpha)\beta_{21} + (80 + 10\alpha) & (15 - 15\alpha)\beta_{22} + (175 + 5\alpha) \end{pmatrix}$$

Step 2. The pair of problems (8)-(9) takes the form

$$Max\, v$$

$$x_1\,((15-15\alpha)\beta_{11}+(175+5\,\alpha))+x_2\,((20-20\alpha)\beta_{21}+(80+10\,\alpha))\geq v$$

$$x_1\,((8-8\alpha)\beta_{12}+(150+6\,\alpha))+x_2\,((15-15\alpha)\beta_{22}+(175+5\,\alpha))\geq v$$

$$x_1+x_2=1,\quad x_i\geq 0, i=\overline{1,2}\,,\ \beta\in T_{1/2}\,.$$

and

$$Min\ \ w$$

$$((15-15\alpha)\beta_{11}+(175+5\,\alpha))y_1+((8-8\alpha)\beta_{12}+(150+6\,\alpha))y_2\leq w$$

$$((20-20\alpha)\beta_{21}+(80+10\,\alpha))y_1+((15-15\alpha)\beta_{22}+(175+5\,\alpha))y_2\leq w$$

$$y_1+y_2=1,\quad y_j\geq 0,\ j=\overline{1,2}\,,\ \beta\in T_{1/2}\,.$$

Using the constraints $x_1+x_2=1$ and $y_1+y_2=1$, we simplify the last pair of problems as follow

$$Max\, v$$

$$(95-5\alpha)x_1+(20-20\alpha)\beta_{21}+(15-15\alpha)x_1\beta_{11}+(-20+20\alpha)x_1\beta_{21}+80+10\alpha\geq v$$

$$(-25+\alpha)x_1+(15-15\alpha)\beta_{22}+(8-8\alpha)x_1\beta_{12}+(-15+15\alpha)x_1\beta_{22}+175+5\alpha\geq v$$

$$0\leq x_1\leq 1,\quad ,\ \beta\in T_{1/2}\,.$$

and

$$Min\, w$$

$$(25-\alpha)\,y_1+(8-8\alpha)\beta_{12}+(15-15\alpha)\,y_1\beta_{11}+(-8+8\alpha)\,y_1\beta_{12}+150+6\alpha\leq w$$

$$(-95+5\alpha)\,y_1+(15-15\alpha)\beta_{22}+(15-15\alpha)\,y_1\beta_{21}+(-15+15\alpha)\,y_1\beta_{22}+175+5\alpha\leq w$$

$$0 \le y_1 \le 1, \quad , \beta \in T_{1/2} .$$

Solving the last pair of problems by Nimbus software Miettinen & Mäkelä (2006), we obtain the following solutions for different values of α

Note that in Tables 1 and 2 for all values of $\alpha \neq 1$, we have $\sum \beta_{ij}$ =2. For α = 1, we do not provide the β_{ij} values, because the game (4) corresponding to (10) reduces to a crisp matrix game (see item (iii) of Remark 3.3 below).

Assume, for instance, that the players have actually chosen the α-cut levels as α_1=1/2, α_2=2/3, respectively. Then α =*Max* {1/2, 2/3}=2/3. The corresponding α-NL solution to the considered game with cut-level 2/3 is $x(2/3) = (0.7743, 0.2257)$,

$$y(2/3) = (0.2568, 0.7432), \quad v(2/3) = 161.52$$

and

$$w(2/3) = 160.32 \text{, (see Tables 1 and 2).}$$

Example 3.2. Assume that in Example 3.1 $\beta_{ij} = 1/2, \;\; i = \overline{1, m}, \; j = \overline{1, n}$, i.e. Nature chooses the ideal neutral behaviour towards players, then the pair of problems (8) and (9) will reduce to the following pair of problems

Max v

$$\begin{aligned} &(1/2)(185 - 5\alpha)x_1 + 90 \ge v \\ &(1/2)(-57 + 9\alpha)x_1 + (1/2)(365 - 5\alpha) \ge v \end{aligned} \tag{11}$$

$$0 \le x_1 \le 1,$$

and

Min w

$$\begin{aligned} &(32 - 8\alpha)y_1 + 158 - 2\alpha \le w \\ &-90y_1 + 190 - 10\alpha \le w \end{aligned} \tag{12}$$

$$0 \le y_1 \le 1.$$

The solution of this pair of problems is

$$\bar{x}(\alpha) = (\frac{185 - 5\alpha}{242 - 14\alpha}, \frac{57 - 9\alpha}{242 - 14\alpha}),$$

$$v(\alpha) = \frac{25\alpha^2 - 4370\alpha - 77785}{484 - 28\alpha} \tag{13}$$

$$\bar{y}(\alpha) = (\frac{57 - 9\alpha}{242 - 4\alpha}, \frac{185 - 5\alpha}{242 - 4\alpha}),$$

and

$$w(\alpha) = \frac{25\alpha^2 - 4370\alpha - 77785}{484 - 28\alpha}. \tag{14}$$

We obtain the following particular solutions, by taking different values of α

Remark 3.3. It is interesting to note the following.

(i) the obtained solution $(x(\alpha), y(\alpha), v(\alpha), w(\alpha))$ is expressed as an explicit function of the cut-level α. Hence if the players change the confidence level α, there is no need for solving the pair of problems (11)-(12) again. This shows some flexibility in our approach and sensitivity analysis with respect to α can be done.

(ii) $v(\alpha) = w(\alpha)$, for all $\alpha \in]0,1]$. This means that what Player I gains is exactly what Player II looses. In other words, the zero-sum game character of the game is maintained. The solution obtained for the game (10) by Bector et al. (2004a) is
for Player I, $x^* = (0.7725, 0.2275)$ and V=160.91;
for Player II $y^* = (0.2275, 0.7725)$ and W= 160.65.

As we see $V \neq W$, the players receive different values. This difference, may be, is due to the fact that the authors have taken different adequacy margins $\tilde{p}_1 = \tilde{p}_2 = (0.08, 0.01, 0.011)$ and $\tilde{q}_1 = \tilde{q}_2 = (0.14, 0.15, 0.17)$ at the defuzzifaction step (see pages 148-149 in Bector & Chandra (2005)).
We notice also that these results are close to ours when the confidence level is $\alpha = 1$:
$x(1) = (15/19, 4/19)$ =(0.7894, 0.2105), $y(1) = (4/19, 15/19)$ = (0.2105, 0.7894) and $v(1) = w(1)$ =161.05. In Bector et al. (2004a) the approach depends on the first Yager's index Yager (1981) given by F($\tilde{d}$) = $\dfrac{\int_{d_L}^{d_U} x\mu(x)\,dx}{\int_{d_L}^{d_U} \mu(x)\,dx}$ (the centroid of $\tilde{d}$), where $\tilde{d}$ is a fuzzy number and $\mu(.)$ is its membership function. The index function F(.) is used along with adequacies $\tilde{p}_1$ and $\tilde{q}_2$ for ranking the fuzzy numbers involved in the constraints of the obtained pair of dual fuzzy linear programming problems. If the players change the index, the solution of the game has to be recalculated.

(iii) When α =1, the fuzzy game (4) corresponding to (10) reduces to the following crisp matrix game

$$A^M = \begin{pmatrix} 180 & 156 \\ 90 & 180 \end{pmatrix} \tag{15}$$

the entries of which are the main values of the fuzzy entries of the game (10). The solution of the game (15) is $\bar{x} = (0.7894, 02105)$, $\bar{y} = (0.2105, 0.7894)$ the value of the game is v =161.05. Replacing α by 1 in (13)-(14), we get the same solution. Indeed, we obtain $x(1) = (0.7894, 0.2105) = (15/19, 4/19)$, $y(1) = (0.2105, 0.7894)$, and $v(1) = w(1)$ =161.05 (see Tables 3 and 4). Moreover, it is clear that when $\alpha \to 1$, the solution (13)-(14) of the fuzzy game (10) tends to the solution of the crisp matrix game (15). Our solution is exactly the same as the one obtained in Vijay et al. (2007), for α =1.

Table 3.

α	$x_1(\alpha)$	$x_2(\alpha)$	$v(\alpha)$
1/3	0.77247	0.22753	160.81
1/2	0.7766	0.2234	160.86
2/3	0.7808	0.2192	160.92
1	0.7894	0.2105	161.05

Table 4.

α	$y_1(\alpha)$	$y_2(\alpha)$	$w(\alpha)$
1/3	0.22753	0.77247	160.81
1/2	0.2234	0.7766	160.86
2/3	0.2192	0.7808	160.92
1	0.2105	0.7894	161.05

(iv) The game (10) has been also treated using the Li's multiobjective approach Li (1999). The following results have been obtained for Player I, $\bar{x}=(15/19, 4/19)$ and $\tilde{v}^*=(v_L^*, v^*, v_U^*)$=(155.0025, 161.05, 164.736);
for Player II, $\bar{y}=(4/19, 15/19)$ and $\tilde{w}^*=(w_L^*, w^*, w_U^*)$=(155.264, 161.05, 171.052).

It is interesting to notice that the optimal strategies $\bar{x}=(15/19, 4/19)$, $\bar{y}=(4/19, 15/19)$ and the main value $v^* = w^*$=161.05 are exactly the same as for the crisp matrix game (15), the entries of which are the main values of the entries of the fuzzy matrix game (10), respectively. Further, by solving the crisp matrix game

$$A^L = \begin{pmatrix} 175 & 150 \\ 80 & 175 \end{pmatrix} \tag{16}$$

the entries of which are the lower bounds of the fuzzy entries of the game (10), we get the value of the game v_L = 155.208, which is between v_L^*=155.0025 and w_L^*=155.264. Next, by solving the crisp matrix game

$$A^U = \begin{pmatrix} 190 & 158 \\ 100 & 190 \end{pmatrix} \quad (17)$$

the entries of which are the upper bounds of the fuzzy entries of the game (10), we get the value of the game v_U =166.393, which is between v_U^* =164.736 and w_U^* =171.052. It would be interesting to find out whether this relation between the proposed solution in Li (1999) and Li & Yang (2004) and the solutions of the crisp matrix games (15)-(17) is true, in general, for matrix games of type (10) (fuzzy matrix games with TFN payoffs), in which case, it makes sense to propose the solution consisting of the optimal strategies of the crisp matrix game corresponding to (15) and the fuzzy value $\tilde{v} = (v_L, v, v_U)$, where v, v_L and v_U are the values of the crisp matrix games corresponding to (15)-(17) respectively, as a solution to the considered game. The advantage of this solution over the Li's would be its computational simplicity, since it reduces to the resolution of three independent crisp matrix games.

The following example is taken from Maeda (2003).

Example 3.3. (Computation of α-NL solution) Consider the fuzzy game defined by the fuzzy payoff matrix

$$\tilde{A} = \begin{pmatrix} 1\tilde{8}0 & 1\tilde{5}6 \\ 9\tilde{0} & 1\tilde{8}0 \end{pmatrix} \quad (18)$$

It is assumed that the entries of $\tilde{A}$ are the following STFNs, $1\tilde{8}0 = (180,5)$, $1\tilde{5}6 = (156,6)$, $9\tilde{0} = (90,10)$. Let us compute an α-NL solution of this game by Procedure 3.1, assuming p = 1/2.

Step1. Ask the players to provide α-cut levels. Assume that they have chosen α_1 and α_2 respectively. Let α =Max{ α_1, α_2 } in order to satisfy the choice of both players.

First we solve the problem in general, then, at the end, we will assume concrete values for α_1 and α_2.

$[\tilde{a}_{ij}]^{\alpha} = \{ a_{ij} / \mu_{\tilde{a}_{ij}}(a_{ij}) \geq \alpha \} = [a_{ij}^{L\alpha}, a_{ij}^{U\alpha}]$, $i = \overline{1,2}$, $j = \overline{1,2}$, then using Remark 3.2, we obtain $[180]^{\alpha} = [185 - 5\alpha, 175 - 5\alpha]$, $[156]^{\alpha} = [150 + 6\alpha, 162 - 6\alpha]$, and $[90]^{\alpha} = [80 + 10\alpha, 100 - 10\alpha]$.

Further, we have $mnp = 2 \times 2 \times 1/2 = 2$, then $T_{1/2} = \left\{ \beta = (\beta_{ij}) / \beta \in T \text{ and } \sum_{i,j} \beta_{ij} = 2 \right\}$.

Compute the matrix $A_{\beta} = (\beta_{ij} (a_{ij}^{U\alpha} - a_{ij}^{L\alpha}) + a_{ij}^{L\alpha})_{1 \leq i \leq m, 1 \leq j \leq n}$, $\beta \in T_{1/2}$. Since the payoffs are STFNs, according to Remark 3.2

$$A_{\beta} = (\beta_{ij} (a_{ij}^{U\alpha} - a_{ij}^{L\alpha}) + a_{ij}^{L\alpha})_{1 \leq i \leq m, 1 \leq j \leq n} =$$

$$A_{\beta} = ((2 - 2\alpha) h_{ij} \beta_{ij} + a_{ij}^{L\alpha})_{1 \leq i \leq m, 1 \leq j \leq n} =$$

$$\begin{pmatrix} (2-2\alpha)5\beta_{11} + (175 + 5\alpha) & (2-2\alpha)6\beta_{12} + (150 + 6\alpha) \\ (2-2\alpha)10\beta_{21} + (80 + 10\alpha) & (2-2\alpha)5\beta_{22} + (175 + 5\alpha) \end{pmatrix}$$

Step 2. Construct and solve the pair of optimization problems corresponding to problems (8)-(9). We have

$Max\, v$

$$x_1((2-2\alpha)5\beta_{11} + (175 + 5\alpha)) + x_2((2-2\alpha)10\beta_{21} + (80 + 10\alpha)) \geq v$$

$$x_1((2-2\alpha)6\beta_{12} + (150 + 6\alpha)) + x_2((2-2\alpha)5\beta_{22} + (175 + 5\alpha)) \geq v$$

$$x_1 + x_2 = 1, \quad x_i \geq 0, \quad i = \overline{1,2}, \ \beta \in T_{1/2}.$$

$Min\ w$

$$((2-2\alpha)5\beta_{11} + (175 + 5\alpha))y_1 + ((2-2\alpha)6\beta_{12} + (150 + 6\alpha))y_2 \leq w$$

$$((2-2\alpha)10\beta_{21} + (80 + 10\alpha))y_1 + ((2-2\alpha)5\beta_{22} + (175 + 5\alpha))y_2 \leq w$$

Table 5.

α	$x_1(\alpha)$	$x_2(\alpha)$	β_{11}	β_{12}	β_{21}	β_{22}	$v(\alpha)$
1/3	0.7638	0.2362	1.0000	0.0000	0.0000	1.0000	159.30
1/2	0.7978	0.2022	0.9999	0.9999	0.0000	0.0000	162.52
2/3	0.7725	0.2275	1.0000	1.0000	0.0000	0.0000	162.62
1	0.7895	0.2105					161.05

Table 6.

α	$y_1(\alpha)$	$y_2(\alpha)$	β_{11}	β_{12}	β_{21}	β_{22}	$w(\alpha)$
1/3	0.2234	0.7786	1.0000	0.0000	0.9999	0.0000	159.08
1/2	0.2185	0.7815	0.9999	0.0000	1.0000	0.0009	159.52
2/3	0.2159	0.7841	1.0000	0.0000	1.0000	0.0000	159.99
1	0.2105	0.7895					161.05

$$y_1 + y_2 = 1, \quad y_j \geq 0, \quad j = \overline{1,2}, \quad \beta \in T_{1/2}.$$

By Nimbus software Miettinen & Mäkelä (2006), we obtain the following solutions for different values of α

Note that in Tables 5 and 6, for all values of $\alpha \neq 1$, we have $\sum \beta_{ij}$ =2. Here also in both tables we do not provide β_{ij} values for α =1, because the game (4) corresponding to (18) reduces to a crisp matrix game.

Remark 3.4. Assuming, for instance, that the players have concretely chosen the cut-levels α_1=1/2, α_2=2/3, respectively. Let α =*Max* {1/2,2/3}=2/3. The corresponding α-NL solution to the considered game with the cut-level 2/3 is

$$x(2/3) = (0.7725, 0.2275), \quad y(2/3) = (0.2159, 0.7841),$$

$$v(2/3) = 162.62 \text{ and } w(2/3) = 159.99.$$

Maeda (2003) has obtained the following optimal strategies for the game (18)

$x^* = (x_1^*, x_2^*) =$ $(\frac{85+10\mu}{108+12\mu}, \frac{23+2\mu}{108+12\mu})$, $y^* = (y_1^*, y_2^*) =$ $(\frac{23+2\lambda}{108+12\lambda}, \frac{85+10\lambda}{108+12\lambda})$, (19) $\mu, \lambda \in [0,1]$, as Nash equilibrium of the bimatrix game

$$BG(\mu, \lambda) = (S^2, S^2, A(\lambda), -A(\mu))$$

where

$$A(\lambda) = \begin{pmatrix} 185-10\lambda & 162-12\lambda \\ 100-20\lambda & 85-10\lambda \end{pmatrix},$$

and

$$A(\mu) = \begin{pmatrix} 185-10\mu & 162-12\mu \\ 100-20\mu & 85-10\mu \end{pmatrix}$$

For $\lambda = \mu$, the game $BG(\mu, \lambda)$ reduces to the crisp matrix (zero-sum) game

$$G(\lambda) = (S^2, S^2, A(\lambda))),$$

Maeda (2003) has also showed that his results are the same as in Compos (1989), when $\lambda = \mu$. However, the approach of Campos cannot be used for $\lambda \neq \mu$ as Maeda pointed out. For the particular case $\lambda = \mu$ =1/2, the crisp matrix game $G(\lambda)$, becomes $G(1/2) = (S^2, S^2, A(1/2)))$, where

$$A(1/2) = A^M = \begin{pmatrix} 180 & 156 \\ 90 & 180 \end{pmatrix} \tag{20}$$

that is, we obtain the matrix the entries of which are the main values of the entries of the fuzzy matrix game (18). On the other hand, when $\alpha = 1$, the game (4) corresponding to (18) reduces to the matrix game (20) as well, hence our solution coincides with Maeda's solution when $\alpha = 1$ in our model and $\lambda = \mu$ =1/2 in Maeda's model. Indeed, when $\alpha = 1$, our solution of the game (18) is given by Tables 5 and 6 $x(1) = (0.7895, 0.2105)$, $y(1) = (0.2105, 0.7895)$, and $v(1) = w(1) = 161.05$. Putting $\lambda = \mu$ =1/2 in (19), we get the same solution. Note that Maeda did not give any game or fuzzy interpretation of the parameters λ, μ involved in his model.

Remark 3.5. Our approach for solving fuzzy matrix games is different from the existing approaches (Bector et al., 2004a; Bector et al., 2004b; Bector & Chandra, 2005; Campos, 1989; Li, 1999; Li & Yang, 2004; Maeda, 2003; Nishizaki and Sakawa, 1995, 1997, 2001; Vijay et al., 2005a, 2005b; Vijay et al., 2007). From a theoretical point of view, we introduce Nature as a third player that represents the uncertainty involved in the game and use the Laplace's principle of insufficient reason to describe the players' beliefs about the behavior of Nature towards them. Nature has no payoff, its strategies vary in the α-cuts of the fuzzy payoffs. We have used the α-cuts to defuzzify the game, and the pair of optimization problems (8)-(9) we obtain is different from that obtained in existing approaches, it contains the additional variable β and its constraints are not linear**.** Compared to the Maeda (2003) and Li (1999) approaches, our approach is more general in the sense that in Maeda (2003) and Li (1999) it is assumed that the payoffs are TFNs (STFNs for the case of Maeda (2003)), while in our approach the payoffs are assumed to be fuzzy intervals with bounded support as defined by Dubois and Prade. Our method differs from that of Campos (1989) and Bector et al. (2004a) in the sense that we do not use any ranking (index) function to defuzzify the game. Moreover, the solution we propose is flexible in the sense that it is related to the cut-level α=max{ α_1, α_2 }, where α_1, α_2 are determined by Player I and Player II respectively**,** while in Campos (1989), Bector et al. (2004a) and Vijay et al (2005b) the solution depends essentially on the Yager's index function chosen. In Nishizaki & Sakawa (1995) a fuzzy goal is defined for the Player I and a new crisp payoff, based on the membership functions of the fuzzy expected payoff and the fuzzy goal, is introduced for this player as a degree of attainment of his fuzzy goal. Our approach differs from the latter approach in the sense that we do not use fuzzy goals and the fuzzy expected payoff of Player I to define our solution. Moreover, from computational point of view, our approach reduces to the resolution of a pair of independent crisp nonlinear programming problems with linear objective function and quite simple nonlinear constraints (quadratic but not involving squared variables), while in Nishizaki & Sakawa (1995) the computation of the solution requires the resolution of a nonlinear programming problem. Finally, compared to Vijay et al. (2007), in our approach we do not use any fuzzy ordering or relation to express the preferences of players over the fuzzy payoffs or fuzzy expected payoffs. Moreover, the initial fuzzy matrix game is transformed into a special crisp three person game.

4. CONCLUSION

In this paper we have presented a new approach for solving matrix two-person zero-sum games with fuzzy payoffs. Our approach differs from the existing ones in the sense that it does not use any ranking of fuzzy numbers, the payoffs are assumed to be fuzzy intervals with bounded support as defined by Dubois and Prade, and it introduces Nature as a third player that expresses the uncertainty involved in the game. The Laplace's principle of insufficient reason is used to represent the beliefs of player about the behavior of Nature towards them. In addition, we provided a quite simple procedure for the computation of the α-NL solution of the game that reduces to solving a pair of independent crisp nonlinear programming problems. Finally, we think that extending this new approach for solving other types of non cooperative games with fuzzy payoffs and/or fuzzy goals may be a worthy direction of research.

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