Whole space case for solution formula of Korteweg type fluid motion in $\mathbb{R}^3$

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Abstract. In this paper we consider the solution formula of linearized diffusive capillary model of Korteweg type without surface tension in three-dimensional Euclidean space $\mathbb{R}^3$ using Fourier transform. Firstly, we construct the matrix of differential operators from the model problem. Then, we apply Fourier transform to the matrix. In the third step, we consider the resolvent problem of model problem. Finally, we find the solution formula of velocity and density by using inverse Fourier transform. For the further research we can consider not only estimating the solution operator families of the Korteweg theory of capillarity but also estimating the optimal decay for solution to the non-linear problem.

1 Introduction

Water can be found in many different forms in daily life, including ice, liquid water, and water vapour. The solid, water, and vapour or gas phases of water are the names given to these various physical states. One might think of boiling water while brewing a cup of tea as an example of water vapour. Instead, we see water steam erupting from the kettle and filling the kitchen. Water vapour, on the other hand, refers to the gaseous state of water in the natural sciences. Despite the imprecision in this common example, we nevertheless gain a general understanding of how water vapourizes, or changes phases from the liquid to the vapour phase. On chilly winter days, we can observe the water easily changing back into tiny drops of water by looking out the kitchen window. Condensation, the phase change from the vapour to the liquid phase, is the main topic.

In the case of making tea, a temperature difference initiates the phase transition: the water vapourizes when heated, and the vapour steam condenses when the cold window surface cools. The two phases of phase transition are under continuous pressure. In general, the phases that matter exists in are determined by both the influences of temperature and pressure. This reliance on pressure is evident in the mountains, where it boils at lower temperature and at the coast. This condition is caused by the reduced pressure at high altitude. In this paper we consider fluids at constant temperature and phase transition between liquids and gases.

Liquids and gases both have ability to flow, in contrast to solids. Together, they make up the fluid class. Their mass densities, however, greatly differ from one another. This phase border enables us to disguise between various phases using mass density, assuming a constant temperature. It makes sense to be interested in the phase boundaries that separate the liquid phase from vapour phase when thinking about a container filled with a fluid. Let us consider the resolvent problem of the compressible Korteweg type model fluid without surface tension which described in the following

$$\begin{cases} \lambda\rho + \operatorname{div} \mathbf{u} = f \quad \text{in } \Omega \\ \lambda\mathbf{u} - \mu\Delta\mathbf{u} - \nu\nabla\operatorname{div} \mathbf{u} - \kappa\nabla\Delta\rho + \gamma\nabla\rho = \mathbf{g} \quad \text{in } \Omega \end{cases} \tag{1}$$

where $\rho = \rho(x,t)$ and $\mathbf{u} = \mathbf{u}(x,t) = \big(u_1(x,t), u_2(x,t), u_3(x,t)\big)^{\mathrm{T}}$ are fluid density and fluid velocity, respectively. The coefficients of μ and ν are viscosity coefficients and $\operatorname{div} \mathbf{u} = \sum_{j=1}^{3} \partial_j u_j$. κ is a given constant such that $\kappa > 0$. $f = f(x)$ and $\mathbf{g} = \mathbf{g}(x) = \big(g_1(x), g_2(x), g_3(x)\big)^{\mathrm{T}}$ are given functions. Meanwhile, λ and γ are eigen value and positif constant, respectively.

Korteweg formulated a constitutive equation of stress tensors in 1901 that includes density gradients to explain fluid capillarity effects. Dunn and Serrin [1] derived a Korteweg stress tensor in the context of rational mechanics by presenting the interstitial operating thermomechanics. The system governing the motion of isothermal compressible viscous fluid of Korteweg type also investigated by [2]. They described a liquid-vapour two-phase flow with phase transition as diffuse interface model.

Furthermore, we provide an overview of the past mathematical research on the Korteweg type model. First, let us concentrate on difficulties with boundaries. A boundary value problem was handled in a weak formulation by Bresch et.al [3] who also put out a number of boundary conditions. Kotschote [4] examined strong solution for the system governing the motion of isothermal compressible viscous fluid of Korteweg type in bounded and exterior domain. He demonstrated the local well-posedness of a linearised system in an L_p setting by proving an optimal regularity result and then combining it with a fixed-point theorem. He also investigated not only Newtonian fluids but also non-Newtonian fluids [5,6]. In 2014, Kotschote studied the asymptotic stability of non-trivial steady state in bounded domain to describe phase transition [7].

Recently, there are many researchers who consider for whole-space case. Hattori and Li [8,9] proved not only local unique existence theorem but also global well-posedness on smooth solutions. Furthermore, Shibata and Murata [10] considered strong solution in $L_p - L_q$framework. Inna et.al [11] considered half-space model problem for a compressible fluid model of Korteweg type with slip boundary conditions. Maryani and Murata [12] studied the compressible Korteweg type with surface tension in half-space. In contrast, Saito [13] investigated same model problem without surface tension. This article considers a compressible of the Korteweg type in $\mathbb{R}^3$ without surface tension in whole space case.

2 Method

Research methodology of this research is review articles. To prove the Theorem 2, first of all, we consider the resolvent problem of the Korteweg type model fluid. Secondly, we apply Fourier transform the Korteweg type. Finally, we can find the solution formula of fluid velocity and fluid density of Korteweg type.

3 Result

3.1 Compressible Fluid of Korteweg Type

The capillary effects and evaporation can be described in partial differential equations (PDE). This PDE known as Korteweg type. Before we state the main theorem, following is definition of Sobolev space

Definition 1. (Adams & Fournier, 2003) [16]
Let $k \in \mathbb{N} \cup \{0\}$ and $p \in [0,1)$, then the Sobolev Space $W_q^m(\Omega)$ is defined by

$$W_q^m(\Omega) := \{\mathbf{u} \in L_q(\Omega) \mid D^\alpha \mathbf{u} \in L_q(\Omega),\ \ \forall \alpha \text{ with } |\alpha| \leq m\}.$$

The following Theorem is the main result of this article:
Theorem 2. Let $\rho(x,t)$ and $\mathbf{u}(x,t)$ be a pressure and velocity in 3-dimensional Euclidean space, respectively. Let $\{\lambda_j(\xi)\}_{j=1}^4$ be the roots of $\det\left(\lambda\mathbf{I} + \widehat{\mathbf{A}}(\xi)\right) = 0$ where $\lambda_1(\xi) = \lambda_2(\xi) = -\mu|\xi|^2$. Then, for $\lambda_j(\xi)$, $j = 3,4$, we have the following assertions:

1) For $|\xi| \geq \frac{2\sqrt{\gamma}}{\sqrt{(\mu+\nu)^2-4\kappa}}$, we have $\lambda_j(\xi)$, $j = 3,4$ as follows:

$$\lambda_3(\xi) = -\frac{1}{2}(\mu+\nu)|\xi|^2 + \frac{1}{2}|\xi|\sqrt{(\mu+\nu)^2|\xi|^2 - 4(\kappa|\xi|^2+\gamma)}$$
$$\lambda_4(\xi) = -\frac{1}{2}(\mu+\nu)|\xi|^2 - \frac{1}{2}|\xi|\sqrt{(\mu+\nu)^2|\xi|^2 - 4(\kappa|\xi|^2+\gamma)}.$$

2) For $|\xi| \leq \frac{2\sqrt{\gamma}}{\sqrt{(\mu+\nu)^2-4\kappa}}$, we have $\lambda_j(\xi)$, $j = 3,4$ as follows:

$$\lambda_3(\xi) = \overline{\lambda_4(\xi)} = -\frac{1}{2}(\mu+\nu)|\xi|^2 + \frac{i}{2}|\xi|\sqrt{(\mu+\nu)^2|\xi|^2 - 4(\kappa|\xi|^2+\gamma)}.$$

3) For $|\xi| \neq \frac{2\sqrt{\gamma}}{\sqrt{(\mu+\nu)^2-4\kappa}}$, we have the solution formula of $\hat{\rho}(\xi,t)$ and $\hat{\mathbf{u}}(\xi,t)$ as follows:

$$\hat{\rho}(\xi,t) = \left(\frac{\lambda_3(\xi)e^{\lambda_4(\xi)t} - \lambda_4 e^{\lambda_3(\xi)t}}{\lambda_3(\xi)-\lambda_4(\xi)}\right)\hat{\rho}_0(\xi) - i\xi\left(\frac{e^{\lambda_3(\xi)t} - e^{\lambda_4(\xi)t}}{\lambda_3(\xi)-\lambda_4(\xi)}\right)\hat{\mathbf{u}}_0(\xi)$$

and

$$\hat{\mathbf{u}}(\xi,t) = -i\xi(\kappa|\xi|^2 + \gamma)\left(\frac{e^{\lambda_3(\xi)t} - e^{\lambda_4(\xi)t}}{\lambda_3(\xi)-\lambda_4(\xi)}\right)\hat{\rho}_0(\xi) + e^{-\mu|\xi|^2 t}\hat{\mathbf{u}}_0(\xi) + \left(\frac{\lambda_3(\xi)e^{\lambda_4(\xi)t} - \lambda_4 e^{\lambda_3(\xi)t}}{\lambda_3(\xi)-\lambda_4(\xi)} - e^{-\mu|\xi|^2 t}\right)\frac{\xi\xi^T}{|\xi|^2}\hat{\mathbf{u}}_0(\xi).$$

In the following section, we state the steps of the proof. Proving the main Theorem 2, first of all, we introduce a reduced system for (1), then calculate representation formulas for solutions of the reduced system by using the Fourier transform and its inverse transform. Finally, we prove our main theorem for the reduced system (1).

3.2 The proof of Theorem 2

In this subsection, we consider the proof of Theorem 2. Firsly, we can write the time differential of equation (1) in the following

$$\begin{cases} \rho_t + \operatorname{div}\mathbf{u} = f, \\ \mathbf{u}_t - \mu\Delta\mathbf{u} - \nu\nabla\operatorname{div}\mathbf{u} - \kappa\nabla\Delta\rho + \gamma\nabla\rho = \mathbf{g}, . \end{cases} \quad (2)$$

in $[0,\infty) \times \mathbb{R}^3$. The equation (2) can be written in the following form

$$\mathbb{U}_t + \mathbf{A}\mathbb{U} = \mathbb{F} \quad \text{in } \Omega_t \quad (3)$$

with,

$$\mathbb{U} = \begin{pmatrix} \rho \\ \mathbf{u} \end{pmatrix} = \begin{pmatrix} \rho \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & \operatorname{div} \\ -\kappa\nabla\Delta + \gamma\nabla & -\mu\Delta - \nu\nabla\operatorname{div} \end{bmatrix},$$

and $\mathbb{F} = \begin{pmatrix} f \\ \mathbf{g} \end{pmatrix}$. Moreover, the linearise form of equation (4) follows

$$\mathbb{U}_t + \mathbf{A}\mathbb{U} = \mathbf{0} \quad \text{in } \Omega. \quad (4)$$

If we write the equation (4) in the matrix form, we have

$$\begin{bmatrix}\rho_t\\ \mathbf{u}_t\end{bmatrix}\begin{bmatrix}0 & \mathrm{div}\\ -\kappa\nabla\Delta+\gamma\nabla & -\mu\Delta-\nu\nabla\,\mathrm{div}\end{bmatrix}\begin{bmatrix}\rho\\ \mathbf{u}\end{bmatrix}=\begin{bmatrix}0\\ \mathbf{0}\end{bmatrix}$$

$$\begin{bmatrix}\rho_t\\ \mathbf{u}_t\end{bmatrix}+\begin{bmatrix}0 & \sum_{k=1}^{3}\frac{\partial}{\partial x_k}\\ -\kappa\frac{\partial}{\partial x_j}\sum_{k=1}^{3}\frac{\partial^2}{\partial x_k^2}+\gamma\frac{\partial}{\partial x_j} & -\mu\sum_{k=1}^{3}\frac{\partial^2}{\partial x_k^2}-\nu\frac{\partial}{\partial x_j}\sum_{k=1}^{3}\frac{\partial}{\partial x_k}\end{bmatrix}\begin{bmatrix}\rho\\ \mathbf{u}\end{bmatrix}=\begin{bmatrix}0\\ \mathbf{0}\end{bmatrix}$$

for $t>0$, $\mathbb{U}|_{t=0}=\mathbb{U}_0$. Moreover, equation (4) has general solution i.e

$$\mathbb{U}(t)=e^{\mathbf{A}t}C.$$

with initial condition $\mathbb{U}|_{t=0}=\mathbb{U}_0$, then

$$\mathbb{U}(0)=C.$$

Therefore, we have

$$\mathbb{U}(t)=e^{\mathbf{A}t}\mathbb{U}_0.$$

Furthermore, the resolvent problem of equation (4) then applying Fourier transform, we have

$$\lambda\,\widehat{\mathbb{U}}(\xi)+\widehat{\mathbf{A}}(\xi)\widehat{\mathbb{U}}(\xi)=\widehat{\mathbb{F}}(\xi)$$
$$[\lambda\mathbf{I}+\widehat{\mathbf{A}}(\xi)]\widehat{\mathbb{U}}(\xi)=\widehat{\mathbb{F}}(\xi)$$
$$\widehat{\mathbb{U}}(\xi)=[\lambda\mathbf{I}+\widehat{\mathbf{A}}(\xi)]^{-1}\widehat{\mathbb{F}}(\xi)$$
$$\mathbb{U}=\mathcal{F}_x^{-1}\left\{[\lambda\mathbf{I}+\widehat{\mathbf{A}}(\xi)]^{-1}\widehat{\mathbb{F}}(\xi)\right\}$$

where $\det[\lambda\mathbf{I}+\widehat{\mathbf{A}}(\xi)]\neq 0$, $\mathbf{I}$ is an identity matrix and $[\lambda\mathbf{I}+\widehat{\mathbf{A}}(\xi)]^{-1}$ is inverse of $[\lambda\mathbf{I}+\widehat{\mathbf{A}}(\xi)]$. By using adjoint method, firstly, we have determine inverse of $[\lambda\mathbf{I}+\widehat{\mathbf{A}}(\xi)]$ that is

$$[\lambda\mathbf{I}+\widehat{\mathbf{A}}(\xi)]^{-1}=\frac{1}{\det[\lambda\mathbf{I}+\mathbf{A}(\xi)]}\mathrm{adj}[\lambda\mathbf{I}+\widehat{\mathbf{A}}(\xi)].$$

By the properties of Fourier transform for $\widehat{\mathbf{A}}(\xi)$ matrix, we have

$$\widehat{\mathbf{A}}(\xi)=\begin{bmatrix}0 & i\xi_1 & i\xi_2 & i\xi_3\\ i\xi_1(\kappa|\xi|^2+\gamma) & \mu|\xi|^2+\nu\xi_1^2 & \nu\xi_1\xi_2 & \nu\xi_1\xi_3\\ i\xi_2(\kappa|\xi|^2+\gamma) & \nu\xi_2\xi_1 & \mu|\xi|^2+\nu\xi_2^2 & \nu\xi_2\xi_3\\ i\xi_3(\kappa|\xi|^2+\gamma) & \nu\xi_3\xi_1 & \nu\xi_3\xi_2 & \mu|\xi|^2+\nu\xi_3^2\end{bmatrix} \quad (5)$$

with $|\xi|^2=\xi_1^2+\xi_2^2+\xi_3^2$ dan $i=\sqrt{-1}$. Moreover $[\lambda\mathbf{I}+\widehat{\mathbf{A}}(\xi)]$ yields

$$[\lambda\mathbf{I}+\widehat{\mathbf{A}}(\xi)]=\begin{bmatrix}\lambda & i\xi_1 & i\xi_2 & i\xi_3\\ i\xi_1(\kappa|\xi|^2+\gamma) & \lambda+\mu|\xi|^2+\nu\xi_1^2 & \nu\xi_1\xi_2 & \nu\xi_1\xi_3\\ i\xi_2(\kappa|\xi|^2+\gamma) & \nu\xi_2\xi_1 & \lambda+\mu|\xi|^2+\nu\xi_2^2 & \nu\xi_2\xi_3\\ i\xi_3(\kappa|\xi|^2+\gamma) & \nu\xi_3\xi_1 & \nu\xi_3\xi_2 & \lambda+\mu|\xi|^2+\nu\xi_3^2\end{bmatrix} \quad (6)$$

Moreover, we find the determinant of equation (6), that is

$$\det[\lambda\mathbf{I}+\widehat{\mathbf{A}}(\xi)]=\hat{a}_{11}\,\hat{C}_{11}+\hat{a}_{12}\,\hat{C}_{12}+\hat{a}_{13}\,\hat{C}_{13}+\hat{a}_{14}\,\hat{C}_{14}$$
$$=\lambda\,|\widehat{\mathbb{M}}_{11}|-i\xi_1\,|\widehat{\mathbb{M}}_{12}|+i\xi_2\,|\widehat{\mathbb{M}}_{13}|-i\xi_3\,|\widehat{\mathbb{M}}_{14}|.$$

Next, we consider $|\widehat{\mathbb{M}}_{11}|$ which explaining in the following

$$|\widehat{\mathbb{M}}_{11}|=\begin{vmatrix}\lambda+\mu|\xi|^2+\nu\xi_1^2 & \nu\xi_1\xi_2 & \nu\xi_1\xi_3\\ \nu\xi_2\xi_1 & \lambda+\mu|\xi|^2+\nu\xi_2^2 & \nu\xi_2\xi_3\\ \nu\xi_3\xi_1 & \nu\xi_3\xi_2 & \lambda+\mu|\xi|^2+\nu\xi_3^2\end{vmatrix} \quad (7)$$
$$=(\lambda+\mu|\xi|^2)^2\{\lambda+(\mu+\nu)|\xi|^2\}.$$

Similar technique, we can find $|\widehat{\mathbb{M}}_{12}|$, $|\widehat{\mathbb{M}}_{13}|$, and $|\widehat{\mathbb{M}}_{14}|$, we have

$$|\widehat{\mathbb{M}}_{12}|=(i\xi_1)\{(\kappa|\xi|^2+\gamma)(\lambda+\mu|\xi|^2)^2\}$$
$$|\widehat{\mathbb{M}}_{13}|=(-i\xi_2)\{(\kappa|\xi|^2+\gamma)(\lambda+\mu|\xi|^2)^2\}$$
$$|\widehat{\mathbb{M}}_{14}|=(i\xi_3)\{(\kappa|\xi|^2+\gamma)(\lambda+\mu|\xi|^2)^2\}$$

Substituting $|\widehat{\mathbb{M}}_{11}|$, $|\widehat{\mathbb{M}}_{12}|$, $|\widehat{\mathbb{M}}_{13}|$, and $|\widehat{\mathbb{M}}_{14}|$ to (5), we have

$$\det[\lambda\mathbf{I}+\widehat{\mathbf{A}}(\xi)]=(\lambda+\mu|\xi|^2)^2\{\lambda^2+(\mu+\nu)|\xi|^2\lambda+(\kappa|\xi|^2+\gamma)|\xi|^2\}. \quad (8)$$

Furthermore, we are determined an adjoint matrix of $[\lambda\mathbf{I}+\widehat{\mathbf{A}}(\xi)]$ for other minors. Same method with (7), we have

$$|\widehat{\mathbb{M}}_{21}|=(i\xi_1)(\lambda+\mu|\xi|^2)^2$$
$$|\widehat{\mathbb{M}}_{22}|=(\lambda+\mu|\xi|^2)\{\lambda(\lambda+\mu|\xi|^2)+(\xi_2^2+\xi_3^2)[\lambda\nu+(\kappa|\xi|^2+\gamma)]\}$$
$$|\widehat{\mathbb{M}}_{23}|=|\widehat{\mathbb{M}}_{32}|=(\lambda+\mu|\xi|^2)\{(\xi_1\xi_2)[\lambda\nu+(\kappa|\xi|^2+\gamma)]\}$$
$$|\widehat{\mathbb{M}}_{24}|=|\widehat{\mathbb{M}}_{42}|=-(\lambda+\mu|\xi|^2)\{(\xi_1\xi_3)[\lambda\nu+(\kappa|\xi|^2+\gamma)]\}$$
$$|\widehat{\mathbb{M}}_{31}|=-(i\xi_2)(\lambda+\mu|\xi|^2)^2$$
$$|\widehat{\mathbb{M}}_{33}|=(\lambda+\mu|\xi|^2)\{\lambda(\lambda+\mu|\xi|^2)+(\xi_1^2+\xi_3^2)[\lambda\nu+(\kappa|\xi|^2+\gamma)]\}$$
$$|\widehat{\mathbb{M}}_{34}|=|\widehat{\mathbb{M}}_{43}|=(\lambda+\mu|\xi|^2)\{(\xi_2\xi_3)[\lambda\nu+(\kappa|\xi|^2+\gamma)]\}$$
$$|\widehat{\mathbb{M}}_{41}|=(i\xi_3)\{(\kappa|\xi|^2+\gamma)(\lambda+\mu|\xi|^2)^2\}$$
$$|\widehat{\mathbb{M}}_{44}|=(\lambda+\mu|\xi|^2)\{\lambda(\lambda+\mu|\xi|^2)+(\xi_1^2+\xi_2^2)[\lambda\nu+(\kappa|\xi|^2+\gamma)]\}.$$

Based on minor results, we have

$$\hat{C}_{11}=(\lambda+\mu|\xi|^2)^2\{\lambda+(\mu+\nu)|\xi|^2\}$$
$$\hat{C}_{1n}=-(i\xi_{n-1})\{(\kappa|\xi|^2+\gamma)(\lambda+\mu|\xi|^2)^2\}$$
$$\hat{C}_{m1}=-(i\xi_{m-1})(\lambda+\mu|\xi|^2)^2$$
$$\hat{C}_{mn}=(\lambda+\mu|\xi|^2)\{\lambda(\lambda+\mu|\xi|^2)\delta_{mn}+(\delta_{mn}|\xi|^2-\xi_{m-1}\xi_{n-1})[\lambda\nu+(\kappa|\xi|^2+\gamma)]\}$$

where $\delta_{mn}=1$ for $m=n$, $\delta_{mn}=0$ for $m\neq n$, and $m,n=2,3,4$.

Next step, we determine eigen values. According to equation (8), eigen values can be found from $\det[\lambda\mathbf{I}+\widehat{\mathbf{A}}(\xi)]=0$. Therefore,

$$(\lambda+\mu|\xi|^2)^2\{\lambda^2+(\mu+\nu)|\xi|^2\lambda+(\kappa|\xi|^2+\gamma)|\xi|^2\}=0. \quad (9)$$

From equation (9), we have $(\lambda+\mu|\xi|^2)^2=0$ or $\{\lambda^2+(\mu+\nu)|\xi|^2\lambda+(\kappa|\xi|^2+\gamma)|\xi|^2\}=0$. First of all, for $(\lambda+\mu|\xi|^2)^2=0$, we have $\lambda_1=\lambda_2=-\mu|\xi|^2$. Whilst, for $\{\lambda^2+(\mu+\nu)|\xi|^2\lambda+(\kappa|\xi|^2+\gamma)|\xi|^2\}=0$, by using following formula,

$$\lambda_{3,4}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

we have,

$$\lambda_{3,4}=-\frac{1}{2}|\xi|^2(\mu+\nu)\pm\frac{1}{2}|\xi|\sqrt{(\mu+\nu)^2|\xi|^2-4(\kappa|\xi|^2+\gamma)}. \quad (9)$$

On the other hand, according to equation (9), for $|\xi|\geq\frac{2\sqrt{\gamma}}{\sqrt{(\mu+\nu)^2-4\kappa}}$, we have

$$\lambda_3(\xi)=-\frac{1}{2}(\mu+\nu)|\xi|^2+\frac{1}{2}|\xi|\sqrt{(\mu+\nu)^2|\xi|^2-4(\kappa|\xi|^2+\gamma)}$$
$$\lambda_4(\xi)=-\frac{1}{2}(\mu+\nu)|\xi|^2-\frac{1}{2}|\xi|\sqrt{(\mu+\nu)^2|\xi|^2-4(\kappa|\xi|^2+\gamma)}.$$

Moreover, for $|\xi|\leq\frac{2\sqrt{\gamma}}{\sqrt{(\mu+\nu)^2-4\kappa}}$, we have

$$\lambda_3(\xi)=\overline{\lambda}_4(\xi)=-\frac{1}{2}(\mu+\nu)|\xi|^2+\frac{i}{2}|\xi|\sqrt{(\mu+\nu)^2|\xi|^2-4(\kappa|\xi|^2+\gamma)} \quad (10)$$

3.3 Fourier transform of $\widehat{\rho}$ and $\widehat{\mathbf{u}}$

Finding a formula of $\hat{\rho}$ and $\hat{\mathbf{u}}$ which are fluid density and fluid velocity, respectively, we transform $\mathbf{u}$ and ρ by using Fourier transform. First of all, applying div to the second line of equation (3), we have

$$\frac{\partial}{\partial t}\mathbf{u} - \mu\Delta \text{div}\,\mathbf{u} - \nu\Delta \text{div}\,\mathbf{u} - \kappa\Delta\Delta\rho + \gamma\Delta\rho = 0. \quad (11)$$

Let $\varphi = \text{div}\,\mathbf{u}$, then $\frac{\partial}{\partial_t}\text{div}\,\mathbf{u} = \varphi_t$ and equation (11) can be written in the following

$$\varphi_t - \mu\Delta\varphi - \nu\Delta\varphi - \kappa\Delta\Delta\rho + \gamma\Delta\rho = 0. \quad (12)$$

Furthermore, according to first line of equation (3), we have

$$\rho_t = -\varphi \quad (13)$$

Then, by differentiating equation (13) respect to t and substituting to (12), we have

$$\rho_{tt} - (\mu\Delta + \nu\Delta)\rho_t - (-\kappa\Delta\Delta + \gamma\Delta)\rho = 0. \quad (14)$$

Applying equation (14) by Fourier transform, we have

$$\hat{\rho}_{tt} + \alpha|\xi|^2\hat{\rho}_t - (-\kappa|\xi|^2 + \gamma)|\xi|^2\hat{\rho} = 0, \quad (15)$$

where $\alpha = \mu + \nu$, and initial data

$$\hat{\rho}(\xi,0) = \hat{\rho}_0(\xi), \hat{\rho}_t(\xi,0) = -i\xi\hat{\mathbf{u}}_0(\xi). \quad (16)$$

Furthermore, we determine eigen values λ. According to equation (15), we have

$$\lambda_{3,4} = -\frac{1}{2}(\mu+\nu)|\xi|^2 \pm \frac{1}{2}|\xi|\sqrt{(\mu+\nu)^2|\xi|^2 - 4(\kappa|\xi|^2+\gamma)}. \quad (17)$$

By equation (17), we have general solution

$$\hat{\rho}(\xi,t) = c_3 e^{\lambda_3(\xi)t} + c_4 e^{\lambda_4(\xi)t} \quad (18)$$

Substituting equations (16) and (17) to equation (18), we have

$$c_3 = \frac{-i\xi\hat{\mathbf{u}}_0(\xi) - \hat{\rho}_0(\xi)\lambda_4(\xi)}{\lambda_3(\xi)-\lambda_4(\xi)}, c_4 = \frac{\hat{\rho}_0(\xi)\lambda_3(\xi) + i\xi\hat{\mathbf{u}}_0(\xi)}{\lambda_3(\xi)-\lambda_4(\xi)}. \quad (19)$$

Moreover, by substituting equation (19) to (18), yield

$$\hat{\rho}(\xi,t) = \left(\frac{\lambda_3(\xi)e^{\lambda_4(\xi)t} - \lambda_4(\xi)e^{\lambda_3(\xi)t}}{\lambda_3(\xi)-\lambda_4(\xi)}\right)\hat{\rho}_0(\xi) - i\xi\left(\frac{e^{\lambda_4(\xi)t} - e^{\lambda_3(\xi)t}}{\lambda_3(\xi)-\lambda_4(\xi)}\right)\hat{\mathbf{u}}_0(\xi).$$

Next step, we consider the solution formula of $\hat{\mathbf{u}}(\xi,t)$. By applying Fourier transform to second line of equation (2), we have

$$\hat{\mathbf{u}}_t + \mu|\xi|^2\hat{\mathbf{u}} + \nu\xi_j\sum_{k=1}^{3}\xi_k\hat{u}_k + i\kappa\xi_j|\xi|^2\hat{\rho} + i\gamma\xi_j\hat{\rho} = 0. \quad (20)$$

for $j = 1,2,3$. We can write equation (20) to be

$$\hat{\mathbf{u}}_t = (\mu|\xi|^2\mathbb{I} - \nu\xi\xi^T)\hat{\mathbf{u}} - i\xi(\kappa|\xi|^2+\gamma)\hat{\rho}.$$

Since, $\hat{\mathbf{u}}(\xi,t)$ is parallel and orthogonal to ξ, then $\hat{\mathbf{u}}(\xi,t)$ can be written as

$$\hat{\mathbf{u}}(\xi,t) = v(\xi,t)\frac{\xi}{|\xi|} + w(\xi,t) \quad (21)$$

with $v(\xi,t) = \hat{\mathbf{u}}(\xi,t)\frac{\xi}{|\xi|}$ is a scalar and $w(\xi,t)$ is ortogonal to ξ. Furthermore, differentiating equation (21) respect to t, we have

$$\hat{\mathbf{u}}_t(\xi,t) = v_t(\xi,t)\frac{\xi}{|\xi|} + w_t(\xi,t). \quad (22)$$

Substituting (21) to (20), we have

$$\hat{\mathbf{u}}_t(\xi,t) = -v(\xi,t)\left(\mu\xi|\xi| + \nu\frac{\xi\cdot\xi\xi^T}{|\xi|}\right) - w(\xi,t)(\mu|\xi|^2 + \nu\xi\xi^T) - i\xi(\kappa|\xi|^2+\gamma)\hat{\rho}. \quad (23)$$

By substitute equation (22) to the right-hand side (RHS) of equation (23), we have

$$v_t(\xi,t) = -\alpha|\xi|^2 v(\xi,t) - i|\xi|(\kappa|\xi|^2+\gamma)\hat{\rho},\ w_t(\xi,t) = -\mu|\xi|^2 w(\xi,t). \quad (24)$$

By using integration by part to equation (24) for $v_t(\xi,t)$, we have

$$v(\xi,t) = e^{-\alpha|\xi|^2 t}\left[v(\xi,0) - i|\xi|(\kappa|\xi|^2 + \gamma)\int_0^t e^{\alpha|\xi|^2\eta}\hat{\rho}(\xi,\eta)\,d\eta\right] \quad (25)$$

with $v(\xi,0)$ is a constant. Then we can find the result of (25) for $e^{\alpha|\xi|^2\eta}\hat{\rho}(\xi,\eta)$ with $\lambda_{3,4}(\xi) + \alpha|\xi|^2 = -\lambda_{4,3}(\xi)$ and $\lambda_3(\xi)\lambda_4(\xi) = (\kappa|\xi|^2+\gamma)|\xi|^2$, we have

$$e^{\alpha|\xi|^2\eta}\hat{\rho}(\xi,\eta) = \left(\frac{\lambda_3(\xi)e^{-\lambda_3(\xi)\eta} - \lambda_4(\xi)e^{-\lambda_4(\xi)\eta}}{\lambda_3(\xi)-\lambda_4(\xi)}\right)\hat{\rho}_0(\xi) - i\xi\left(\frac{e^{-\lambda_4(\xi)\eta} - e^{-\lambda_3(\xi)\eta}}{\lambda_3(\xi)-\lambda_4(\xi)}\right)\hat{\mathbf{u}}_0(\xi). \quad (26)$$

Integrating equation (26) for $0 \le \eta \le t$ to RHS and LHS, we have

$$\int_0^t e^{\alpha|\xi|^2\eta}\hat{\rho}(\xi,\eta)\,d\eta = \left(\frac{-e^{-\lambda_3(\xi)t} + e^{-\lambda_4(\xi)t}}{\lambda_3(\xi)-\lambda_4(\xi)}\right)\hat{\rho}_0(\xi) + \left(\frac{i\xi}{(\kappa|\xi|^2+\gamma)|\xi|^2}\right)\left(\frac{-\lambda_4(\xi)e^{-\lambda_3(\xi)t} + \lambda_3(\xi)e^{-\lambda_4(\xi)t}}{\lambda_3(\xi)-\lambda_4(\xi)}\right)\hat{\mathbf{u}}_0(\xi) \quad (27)$$

According to equation (27), we have equation (25) with $v(\xi,0) = 0$, and $v(\xi,0)e^{\alpha|\xi|^2\eta} = 0$, then

$$v(\xi,t) = -i|\xi|(\kappa|\xi|^2+\gamma)\left(\frac{e^{\lambda_3(\xi)t} - e^{\lambda_4(\xi)t}}{\lambda_3(\xi)-\lambda_4(\xi)}\right)\hat{\rho}_0(\xi) + \frac{\xi}{|\xi|}\left(\frac{\lambda_3(\xi)e^{\lambda_3(\xi)t} - \lambda_4(\xi)e^{\lambda_4(\xi)\eta}}{\lambda_3(\xi)-\lambda_4(\xi)}\right)\hat{\mathbf{u}}_0(\xi) \quad (28)$$

Moreover, based on equation (23) for $w(\xi,t)$ and initial data $w(\xi,0) = \left(\mathbf{I} - \frac{\xi\xi^T}{|\xi|^2}\right)\hat{\mathbf{u}}_0(\xi)$, we have

$$w(\xi,t) = e^{-\mu|\xi|^2 t}\left(\mathbf{I} - \frac{\xi\xi^T}{|\xi|^2}\right)\hat{\mathbf{u}}_{\mathbf{0}}(\xi) \quad (29)$$

Finally, by substituting equation (28) and (29) to (21), we have

$$\hat{\mathbf{u}}(\xi,t) = -i\xi(\kappa|\xi|^2+\gamma)\left(\frac{e^{\lambda_3(\xi)t} - e^{\lambda_4(\xi)t}}{\lambda_3(\xi)-\lambda_4(\xi)}\right)\hat{\rho}_0(\xi) + e^{-\mu|\xi|^2 t}\hat{\mathbf{u}}_0(\xi) + \left(\frac{\lambda_3(\xi)e^{\lambda_3(\xi)t} - \lambda_4 e^{\lambda_4(\xi)t}}{\lambda_3(\xi)-\lambda_4(\xi)} - e^{-\mu|\xi|^2 t}\right)\frac{\xi\xi^T}{|\xi|^2}\hat{\mathbf{u}}_0(\xi)$$

This proved the Theorem 2.

4 Discussion

In this article, we investigate a result concerning the solution formula of equation system (1) in 3dimensional case. As we known that the example of Korteweg system is cavitation process. There are two ways to represent the vapor-liquid interface that are sharp-interface and diffuse-interface. The diffuseinterface is used to study cavitation.

5 Conclusion

Cavitation as an example of Korteweg type is delivering in this article. This phenomenon is described in PDE. The first equation system of (1) called conservation of mass and the second one is conservation of momentum. This study investigates the solution formula of Korteweg type without surface tension by using Fourier transformation. The solution formula of equation system (1) is formed by the multiplier. Using Weis's Theorem, we can estimate the multipliers.

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