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On a convex combination of Lotka–Volterra operators

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ABSTRACT

We consider a convex combination of two classes of Lotka–Volterra operators defined on 2-dimensional simplex. Earlier, the dynamics of a particular case of the considered operators has been investigated. However, its bijective property was not studied. In this paper, we are able to establish that such maps are homeomorphism of the simplex.

1. Introduction

The simplest non-linear stochastic operators, the quadratic stochastic operators (QSO) were introduced and studied by Bernstein in his study of Theory of Heredity [1]. Since then, an extensive study on QSO [2] and later cubic stochastic operators (CSO) [3,4], have been investigated. Their applications in various fields of natural sciences appeared in [5–10]. Mostly in QSO and CSO, a class of Lotka–Volterra (LV) operators have been intensively explored (see also [11]). We notice that Lotka–Volterra models have been reported to being used as tools for applications in ecological system such as harvesting problem [8]. However, it is not our intent to study the dynamics of a discretised Lotka–Volterra model in this paper. Instead, we investigate one of its properties, bijectivity; which will prove the existence of its inverse. Nevertheless, if we have the Lyapunov function of a bijective operator, this will aid in studying the limiting behaviour of its negative trajectory.

We point out that while surjectivity of any LV stochastic operator was proven [12], the bijectivity of the same class of operator was only proven for quadratic case [2]. In general, the bijectivity of LV stochastic operators is not yet established.

In this paper, we are going to establish the bijectivity of a class of LV stochastic operator which is a convex combination of two different classes of LV operator defined on 2-dimensional simplex.

Assume that $E = \{1, 2, 3\}$. In this paper we consider 2-dimensional simplex is defined by

$$S^2 = \left\{ \mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_i \geq 0, \forall i \in E, \sum_{i\in E} x_i = 1 \right\}. \tag{1}$$

By $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ we denote the standard basis in $\mathbb{R}^3$. Given $\alpha \subset E$ the set $\Gamma_\alpha = conv\{\mathbf{e}_k\}, k \in \alpha$ represent a face of S^2.

A mapping $\mathcal{K} : \mathbb{R}^3_+ \to \mathbb{R}^3_+$ is called *stochastic* if $\mathcal{K}(S^2) \subset S^2$. In this paper, we always consider stochastic operators.

The *trajectory* (orbit) $\{\mathbf{x}^{(n)}\}, n = 0, 1, 2, \ldots$ of $\mathcal{K}$ for an initial value $\mathbf{x}^{(0)} \in S^2$ is defined by

$$\mathbf{x}^{(n+1)} = \mathcal{K}\left(\mathbf{x}^{(n)}\right) = \mathcal{K}^{n+1}\left(\mathbf{x}^{(0)}\right), \quad n = 0, 1, 2, \ldots$$

Denote by $\omega_{\mathcal{K}}\left(\mathbf{x}^{(0)}\right)$ the set of limit points of the trajectory $\{\mathbf{x}^{(n)}\}_{n=0}^{\infty}$.

Definition 1. The operator $\mathcal{K}$ is called *regular* (or *stable*) if for any initial point $\mathbf{x} \in S^2$, the limit

$$\lim_{n\to\infty} \mathcal{K}^n(\mathbf{x})$$

exists.

Definition 2. Let $\mathcal{K}$ be a stochastic operator on S^2 and let A be a maximal measurable subset of S^2 such that for any $\mathbf{x} \in A$ the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \mathcal{K}^m(\mathbf{x}) \tag{2}$$

does not exist. If $\mu(A) > 0$ with the usual Lebesgue measure μ on S^2, then $\mathcal{K}$ is said to be *non-ergodic*; if $\mu(A) = 0$, then $\mathcal{K}$ is called *ergodic by mod* 0.

Recently, in [13], the dynamics of cubic operator $U_\theta = \theta U_1 + (1-\theta)U_2, \theta \in [0, 1]$ defined on S^2 has been investigated. Here

$$U_1(\mathbf{x}) = \begin{cases} x_1' = x_1(1 + x_1x_2 - x_3^2), \\ x_2' = x_2(1 + x_2x_3 - x_1^2), \\ x_3' = x_3(1 + x_3x_1 - x_2^2), \end{cases} \qquad U_2(\mathbf{x}) = \begin{cases} x_1' = x_1(1 + x_3^2 - x_1x_2), \\ x_2' = x_2(1 + x_1^2 - x_2x_3), \\ x_3' = x_3(1 + x_2^2 - x_3x_1). \end{cases}$$

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It was shown that $\varphi(\mathbf{x}) = x_1x_2x_3$ is a Lyapunov function for U_θ, which is increasing for $\theta < \frac{1}{2}$ and decreasing for $\theta > \frac{1}{2}$. Mainly, it was shown that U_θ has the property of being regular for $\theta < \frac{1}{2}$, or non-ergodic for $\theta > \frac{1}{2}$. The biological interpretation of above model is clear. Consider a population of three species, all species will coexist if $\theta < \frac{1}{2}$. Otherwise if $\theta > \frac{1}{2}$, one of the species will eventually be driven to near extinction.

However, the bijectivity of U_θ was not proven. In the present paper, we are going to consider a more general operator than U_θ and prove its bijectivity. As a particular case, this will also imply the bijectivity of U_θ.

2. Main result

In this section, we investigate the mapping $W_\theta : S^2 \to S^2$ defined by

$$W_\theta(\mathbf{x}) = \begin{cases} x_1' = x_1[1 + (2\theta - 1)(x_1^r x_2 - x_3^{r+1})], \\ x_2' = x_2[1 + (2\theta - 1)(x_2^r x_3 - x_1^{r+1})], \\ x_3' = x_3[1 + (2\theta - 1)(x_3^r x_1 - x_2^{r+1})], \end{cases} \tag{3}$$

where $r > 0$, and parameter $\theta \in [0, 1]$. If $\theta = \frac{1}{2}$, then V_θ is an identity mapping. If $r = 1$, then W_θ reduces to U_θ. Note that (3) is a convex combination $W_\theta = \theta W_1 + (1 - \theta)W_2$, where

$$W_1(\mathbf{x}) = \begin{cases} x_1' = x_1(1 + x_1^r x_2 - x_3^{r+1}), \\ x_2' = x_2(1 + x_2^r x_3 - x_1^{r+1}), \\ x_3' = x_3(1 + x_3^r x_1 - x_2^{r+1}), \end{cases} \quad W_2(\mathbf{x}) = \begin{cases} x_1' = x_1(1 + x_3^{r+1} - x_1^r x_2), \\ x_2' = x_2(1 + x_1^{r+1} - x_2^r x_3), \\ x_3' = x_3(1 + x_2^{r+1} - x_3^r x_1). \end{cases}$$

In this section, we are going to prove the bijectivity of (3). We will show that the determinant of Jacobian of W_θ and $\overline{W_\theta}$, the restriction of W_θ at each faces is greater than zero for any $\mathbf{x}$ within the interior of respective domains.

In the sequel, we always assume that $\theta \neq \frac{1}{2}$. Then, from (3) it follows that

$$J_{W_\theta} = (2\theta - 1)\begin{bmatrix} a_{11} & x_1^{r+1} & -(r+1)x_3^r x_1 \\ -(r+1)x_1^r x_2 & a_{22} & x_2^{r+1} \\ x_3^{r+1} & -(r+1)x_2^r x_3 & a_{33} \end{bmatrix}, \tag{4}$$

where

$$\begin{aligned} a_{11} &= \frac{1}{2\theta - 1} + (r+1)x_1^r x_2 - x_3^{r+1}, \\ a_{22} &= \frac{1}{2\theta - 1} + (r+1)x_2^r x_3 - x_1^{r+1}, \\ a_{33} &= \frac{1}{2\theta - 1} + (r+1)x_3^r x_1 - x_2^{r+1}, \end{aligned} \tag{5}$$

and get its determinant

$$|J_{V_\theta}| = \lambda \begin{vmatrix} \frac{a_{11}}{x_1} & x_1^r & -(r+1)x_3^r \\ -(r+1)x_1^r & \frac{a_{22}}{x_2} & x_2^r \\ x_3^r & -(r+1)x_2^r & \frac{a_{33}}{x_3} \end{vmatrix}, \tag{6}$$

where $\lambda = x_1x_2x_3(2\theta - 1)^3$.

Lemma 1. *Let $(x_1, x_2, x_3) \in S^2$ with $x_k > 0$, $k \in E$ and $r > 0$, then $(r+1)x_i^r x_j - x_k^{r+1} \in (-1, 1)$, for any $i \neq j \neq k$, $i, j, k \in E$.*

Proof. One can see that $x_k \le 1 - x_i - x_j$, hence

$$(r+1)x_i^r x_j - x_k^{r+1} > (r+1)x_i^r x_j - x_k > x_i + x_j + (r+1)x_i^r x_j - 1 > -1.$$

Furthermore, one can show that $(r+1)x_i^r x_j - x_k^{r+1} < (r+1)x_i^r(1 - x_i) - x_k^{r+1}$. Since $x_i^r(1 - x_i)$ has a maximum of $\left(\frac{r}{r+1}\right)^r \frac{1}{r+1}$ at $x_i = \frac{r}{r+1}$, we get

$$(r+1)x_i^r x_j - x_k^{r+1} \le \left(\frac{r}{r+1}\right)^r - x_k^{r+1} \le 1 - x_k^{r+1} < 1, \tag{7}$$

hence $(r+1)x_i^r x_j - x_k^{r+1} \in (-1, 1)$. □

Lemma 2. *Let $\mathbf{x} \in \operatorname{int} S^2$, then $a_{11}, a_{22}, a_{33} > 0$ if $\theta > \frac{1}{2}$, and $a_{11}, a_{22}, a_{33} < 0$ if otherwise.*

Proof. Let us consider a_{11}, the rest can be proceeded by the same argument. Now using (5) together with Lemma 1 one can show that

$$(2\theta - 1)[(r+1)x_1^r x_2 - x_3^{r+1}] \in (-1, 1)$$

for $\theta \in [0, 1]$. Hence,

$$a_{11} = \frac{1 + (2\theta - 1)[(r+1)x_1^r x_2 - x_3^{r+1}]}{(2\theta - 1)x_1}, \tag{8}$$

which is positive for $\theta > \frac{1}{2}$, and negative if $\theta < \frac{1}{2}$. □

Now, we are ready to formulate the main result.

Theorem 1. *Let $\theta \in [0, 1] \setminus \left\{\frac{1}{2}\right\}$, then the mapping W_θ defined by (3) is a homeomorphism of S^2.*

Proof. Let the mapping $\overline{W} : S^1 \to S^1$ be the restriction of W_θ on each faces, $\Gamma_\alpha, \alpha \subset E$. Then the Jacobian under the restriction at each faces is given by

$$J_{\overline{W}} = \begin{bmatrix} a_{ii} & a_{ij} \\ -\tau a_{ij} & a_{jj} \end{bmatrix}, \qquad i, j \in \alpha, \tag{9}$$

where $\tau > 0$. By Lemma 2, we can easily show that determinant $|J_{\overline{W}}| > 0$. Therefore, it is sufficient to show that the mapping W_θ is a local homeomorphism at any interior point of S^2.

Let $\mathbf{x} \in \operatorname{int} S^2$, denoting (6) as $|J_{V_\theta}| = \lambda|A|$, $A = (a_{ij})_{i,j=1}^3$; one could show that

$$\begin{aligned} |A| &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &\quad - a_{11}a_{23}a_{32} - a_{33}a_{12}a_{21} - a_{22}a_{13}a_{31} \\ &= a_{11}a_{22}a_{33} + [1 - (r+1)^3]a_{12}a_{23}a_{31} \\ &\quad + (r+1)(a_{11}a_{23}^2 + a_{33}a_{12}^2 + a_{22}a_{31}^2). \end{aligned}$$

Suppose $\theta > \frac{1}{2}$. Since $2\theta - 1 \in [0, 1)$ we have

$$a_{11} = \frac{1 - (2\theta - 1)x_3^{r+1}}{(2\theta - 1)x_1} + (r+1)x_1^r \frac{x_2}{x_1} > (r+1)x_1^r \frac{x_2}{x_1}. \tag{10}$$

Doing the same for a_{22} and a_{33} we obtain

$$\begin{aligned} |A| &> (r+1)^3 a_{12}a_{23}a_{31} + [1 - (r+1)^3]a_{12}a_{23}a_{31} \\ &\quad + (r+1)(a_{11}a_{23}^2 + a_{33}a_{12}^2 + a_{22}a_{31}^2) \\ &> 0. \end{aligned}$$

Now suppose $\theta < \frac{1}{2}$. By Lemma 2 we have $a_{11}, a_{22}, a_{33} < 0$, and it is clear that $|A| < 0$. Recall that $\lambda = x_1x_2x_3(2\theta - 1)^3$, thus we have the determinant $|J_{V_\theta}| = \lambda|A| > 0$ for any $\theta \in [0, 1] \setminus \left\{\frac{1}{2}\right\}$. This completes the proof. □

The proved Theorem 1 implies the next result.

Corollary 1. *For any initial point $\mathbf{x}^{(0)} \in S^2$, the negative trajectory*

$$\mathbf{x}^{(0)}, W_\theta^{-1}\mathbf{x}^{(0)}, W_\theta^{-2}\mathbf{x}^{(0)}, \dots$$

exists.

Let $r = 1$, then W_θ is reduced to U_θ, and recall that $\varphi(\mathbf{x}) = x_1x_2x_3$ is a decreasing Lyapunov function for $U_\theta, \theta > \frac{1}{2}$.

Theorem 2. *For any initial point $\mathbf{x}^{(0)} \in \operatorname{int} S^2$, the negative trajectory of U_θ exists, and it always converges to $\mathbf{c} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ if $\theta > \frac{1}{2}$.*

Proof. Assume $\mathbf{x}^{(0)} \in \operatorname{int} S^2$. Since $\varphi(\mathbf{x}) = x_1x_2x_3$ is a decreasing Lyapunov function of U_θ, we have

$$\dots \ge \varphi(\mathbf{x}^{(-2)}) \ge \varphi(\mathbf{x}^{(-1)}) \ge \varphi(\mathbf{x}^{(0)}) \ge \varphi(\mathbf{x}') \ge \varphi(\mathbf{x}'') \ge \cdots, \tag{11}$$

i.e. $\lim_{t\to-\infty}\varphi(\mathbf{x}^{(t)}) = \max\{x_1x_2x_3\} = \frac{1}{27}$, which is true only for $x_1 = x_2 = x_3 = \frac{1}{3}$. Hence, the negative trajectory converges to $\mathbf{c}$. □

3. Conclusion

The Bijectivity of the mapping (3) guarantee the existence of its inverse and negative trajectory regardless of its parameter θ. Furthermore, there is no correlation between regularity of (3) and its properties of being a bijection as shown for $r = 1$.

Suppose there is an ecological systems which satisfies the LV operator (3), each evolutionary path taken to achieve a state will be unique, and its ancestral distribution could be traced.

Declaration of competing interest

We wish to confirm that there are no known conflicts of interest associated with this publication and there has been no significant financial support for this work that could have influenced its outcome.

Data availability

No data was used for the research described in the article.

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