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On the approximation of the function on the unite sphere by the spherical harmonics

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ABSTRACT

In this paper we discuss convergence and summability of the Fourier series of distributions in the domains where it coincides with smooth functions in eigenfunction expansions of the Laplace operator on the unite sphere. We consider representation of the distributions defined on the unit sphere by its Fourier-Laplace series by the spherical harmonics in different topologies. Mainly we study the Chesaro method of summation such a series.

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INTRODUCTION

Approximation of a function defined on a unit sphere with the system of orthogonal functions is important in the solutions of the boundary and initial problems for the equation of the mathematical physics, such as heat or wave equations, in a unit ball. An example of such system of orthogonal functions is a system of orthogonal polynomials on a unit sphere.

Spherical harmonics are the restrictions of the space homogeneous harmonic polynomials on a unit sphere S^N, $N \geq 3$. If Y_n and Y_m are spherical harmonics degree n and m, $n \neq m$, respectively then they are orthogonal in $L_2(S^N)$:

$$\int_{S^N} Y_n(x)\, Y_m(x) d\sigma(x) = 0, \quad \mathrm{n} \neq \mathrm{m},$$

here $d\sigma(x)$ is a surface element on a unit sphere.

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A set of all spherical harmonics of degree n is a $a_n = \frac{(N+n)!}{N!n!} - \frac{(N+n-2)!}{N!(N-2)!}$ dimensional subspace of $L_2(S^N)$. Let $\{Y_1^n, Y_2^n, \dots\dots, Y_{a_n}^n\}$ is the orthonormal basis of this subspace. Then union of all such basis will be orthonormal basis in $L_2(S^N)$.

Denote by $P_n^\lambda(t)$ the Gegenbauer polynomials (see in [6]). These polynomials can be generated from the function $(1-2th+h^2)^{-\lambda}$ as the coefficients in expansion of this generating function in the Maclaurin series with respect variable h . Then the following summation formula is valid

$$P_n^{\frac{N-2}{2}}(cos\gamma) = \frac{2\,\pi^{\frac{N}{2}}}{\left(n+\frac{N-2}{2}\right)\Gamma\left(\frac{N-2}{2}\right)} \sum_{j-1}^{a_n} Y_j^n(x) Y_j^n(y),$$

where γ, $0 \le \gamma \le \pi$, is spherical distance between x and y.

Let $f \in L_2(S^N)$. Then by

$$f_j^n = \int_{S^N} f(x) Y_j^n(x) d\sigma(x), \quad j = 1, 2, 3, \dots\dots, a_n$$

denote its Fourier coefficients by the orthonormal system $\{Y_1^n, Y_2^n, \dots\dots, Y_{a_n}^n\}$. Then the Fourier-Laplace series of the function f define as follows

$$f(x) = \sum_{n=1}^{\infty} Y_n^{\frac{N-2}{2}}(f, x), \tag{1}$$

where

$$Y_n^{\frac{N-2}{2}}(f, x) = \sum_{j=1}^{a_n} Y_j^n(x) f_j^n \quad .$$

Then taking into account summation formula above for the Gegenbauer polynomials obtain

$$Y_n^{\frac{N-2}{2}}(f, x) = \frac{\left(n+\frac{N-2}{2}\right)\Gamma\left(\frac{N-2}{2}\right)}{2\,\pi^{\frac{N}{2}}} \int_{S^N} f(y) P_n^{\frac{N-2}{2}}(cos\gamma) \;\; d\sigma(y), \tag{2}$$

Note that for any function $f(y)$ from $L_2(S^N)$ the following energetic equation known also as Parseval's equality is valid

$$||f|| = \sqrt{\sum_{n=1}^{\infty} \left\| Y_n^{\frac{N-2}{2}}(f, x) \right\|^2} \, ,$$

where $||f||$ a norm of the function f in $L_2(S^N)$.

Hence from the Parseval equality it follows that for any function f in $L_2(S^N)$ its Fourier-Laplace series (1) converges in the norm of the space $L_2(S^N)$.

There are many research works and literature devoted to the study of the Fourier-Laplace series on a sphere (1). Here we note a book [6] where author analysed the problems of summability, localization problems and the problems of the almost everywhere convergence. In [6] these problems studied in the Lebesgue spaces $L_p(S^N)$, $p \ge 1$. In [1] author studied the problems of uniform convergence and localization of the Cesaro means of the series (1) in the Nikolskii classes. The problems in the spaces of distributions studied in [2], [3], [4], [5].

THE CESARO MEANS OF THE FOURIER-LAPLACE SERIES.

Let α is a non negative number. Then we define the Cesaro means of order α of partial sums of the Fourier-Laplace series (1) as follows

$$S_n^\alpha f(x) = \frac{1}{A_n^\alpha} \sum_{k=1}^{n} A_{n-k}^\alpha Y_k^{\frac{N-2}{2}}(f, x), \tag{3}$$

where $A_m^\alpha = \frac{\Gamma(\alpha+m+1)}{\Gamma(\alpha+1)m!}$. It is clear that if $\alpha = 0$, then (3) is the partial sum of (1).

Subtitude (2) into (3) and change the order of integration and summation, then obtain

$$S_n^\alpha f(x) = \int\limits_{S^N} f(y)\Theta^\alpha(x,y,n) \ \ d\sigma(y), \qquad (4)$$

where

$$\Theta^\alpha(x,y,n) = \frac{1}{A_n^\alpha}\sum_{k=1}^{n} A_{n-k}^\alpha \frac{\left(n+\frac{N-2}{2}\right)\Gamma\left(\frac{N-2}{2}\right)}{2\,\pi^{\frac{N}{2}}} P_n^{\frac{N-2}{2}}(cos\gamma). \qquad (5)$$

Thus, the Cesaro means of the partial sums of the Fourier-Laplace series (3) we can consider as an integral operator (4) with the kernel (5).

For the kernel (5) the following estimations are valid [1]:

Theorem 1. If $\alpha > -1$ and $\left|\frac{\pi}{2} - \gamma\right| \le \frac{n}{n+1}\cdot\frac{\pi}{2}$, then for $n \to \infty$

$$\Theta^\alpha(x,y,n) = O(1)\cdot\frac{n^{\frac{N-1}{2}-\alpha}}{(2\,sin\,\gamma)^{\frac{N-1}{2}}\cdot\left(2\,sin\frac{\gamma}{2}\right)^{1+\alpha}} +$$

$$+\frac{O\left(n^{\frac{N-1}{2}-\alpha-1}\right)}{(sin\,\gamma)^{\frac{N+1}{2}}\left(sin\frac{\gamma}{2}\right)^{1+\alpha}} + \frac{O\left(\frac{1}{n}\right)}{\left(sin\frac{\gamma}{2}\right)^{1+N}},$$

if $\alpha > -1$ and $0 < \gamma_0 \le \gamma \le \pi$, then for $n > 1$

$$\Theta^\alpha(x,y,n) = O(1)\cdot n^{N-1-\alpha},$$

if $\alpha > -1$ and $0 \le \gamma \le \pi$, then for $n > 1$

$$\Theta^\alpha(x,y,n) = O(1)\cdot n^N$$

THE SPACES OF THE DISTRIBUTIONS ON S^N.

Denote by $C^\infty(S^N)$ a space of infinitely differentiable functions on a sphere S^N. This is a linear vector space and it can be equipped with the topology of the uniform convergence of the sequence of the functions and their derivative of any order and denote this topological space as $\mathcal{E}(S^N)$. Denote by $\mathcal{E}'(S^N)$. the space of the linear continues functionals on $\mathcal{E}(S^N)$.

Note that a set of all polynomials $Y_j^n(x)$ belong to $\mathcal{E}(S^N)$. Then for any functional $f\epsilon\mathcal{E}'(S^N)$ it's the Fourier coefficients defined as action of this functional on the function $Y_j^n(x)$ as follows $f_j^n =< f,\ Y_j^n(x) >$. Then corresponding decomposition of the functional f by orthogonal system of the test functions represented as follows

$$f = \sum_{n=1}^{\infty}\sum_{j=1}^{a_n} Y_j^n(x) f_j^n\,.$$

It is always convergence in the week topology which is a topology of the space $\mathcal{E}'(S^N)$.

MAIN THEOREM

Beside the weak (in $\mathcal{E}'(S^N)$) and strong (in $L_2(S^N)$) topologies mentioned above one can consider other topologies for the equality (?) in the sub domains of S^N where a distribution coincides with the regular function. For example, if a singular distribution coincides with continuous function in $V \subseteq S^N$, then we can study uniform convergence or summability of the series in V.

For the classifications of the singularities of the distributions we use the Sobolev spaces $W_2^\ell(S^N)$ where ℓ any real number. Then $\cap\, W_2^\ell(S^N) = \mathcal{E}'(S^N)$.

The main result of this paper is the following theorem

Theorem 2. *Let in* $V \subseteq S^N$ *is a domain and* f *is a continuous in* V *functional from the space* $W_2^{\ell}(S^N) \cap \mathcal{E}'(S^N)$, $\ell > 1/2$ *. Then series (?) convergence uniformly in any compact set from* V *if* $\alpha \geq \frac{N-1}{2} + \ell$.

PROOF OF THEOREM 2.

Let us bring in a new function $g_1(x)$ which is continuous on the sphere S^N and coincides with the functional f in V and $g_2 = f - g_1(x)$. Then uniformly convergence of $S_n^{\alpha} g_1(x)$ in any compact set from V to f follows from estimations of the kernel (5) in the theorem 1 and the theorem below [1]

Theorem 3. Let in $g(x)$is a continuous function from the Nikolskíi class $H_2^{\ell}(S^N)$ and $\alpha + \ell \geq \frac{N-1}{2}, \ell > 1/2$. Then

$$\lim_{n\to\infty} S_n^{\alpha} g(x) = g(x),$$

uniformly in on S^N.

Note that a functional g_2 is equal to zero in the domain V . Thus, to prove the theorem 2 we need to prove

$$\lim_{n\to\infty} S_n^{\alpha} g_2(x) = 0,$$

which follows from the following theorem on the localization [5]

Theorem 4. Let in $g(x)$ is a functional from the class $H_2^{-\ell}(S^N) \cap \mathcal{E}'(S^N)$ and $\alpha \geq \ell + \frac{N-1}{2}$. Then

$$\lim_{n\to\infty} S_n^{\alpha} g(x) = 0,$$

uniformly in any compact set from $S^N \backslash supp(g)$.

Theorem 2 is proved. ■

CONFLICTS OF INTEREST

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REFERENCES

[1] A.K. Pulatov, On the uniformly convergence and localization of arithmetic means of Fourier-Laplace series. J. Soviet Doclads, 258(3) (1981), 554-556.

[2] A.A. Rakhimov, On the uniform convergence of Fourier series, Malaysian Journal of Mathematical Sciences, 10 (2016), 55–60.

[3] A.A. Rakhimov, On the uniform convergence of Fourier series on a closed domain. Eurasian Mathematical Journal, 8(3) (2017), 60–69.

[4] A.A. Rakhimov, Localization of the Spectral Expansions Associated with the Partial Differential Operators, Springer Book "Mathematical Methods in Engineering", 1 (2019), 217-233.

[5] A.A. Rakhimov, Some problems of summability of spectral expansions connected with Laplace operator on sphere, ArXiv. Math, SP (2009), 1-8.

[6] S.B. Topuriya, The Fourier-Laplace series on a sphere, Tbilisi University Press, 1987.