On the approximation of the function on the unite sphere by the spherical harmonics

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CONTENT

❖ INTRODUCTION

Solution of some boundary value problems and initial problems in unique ball leads to the convergence and summability problems of Fourier series of given function by eigenfunctions of Laplace operator on a sphere - spherical harmonics. Such a series are called as Fourier-Laplace series on sphere. There are a number of works devoted investigation of these expansions in different topologies and for the functions from the various functional spaces. In this paper we discuss convergence and summability of the Fourier series of distributions in the domains where it coincides with smooth functions in eigenfunction expansions of the Laplace operator on the unite sphere. We consider representation of the distributions defined on the unit sphere by its Fourier-Laplace series by the spherical harmonics in different topologies. Mainly we study the Chesaro method of summation such a series.

❖ INTRODUCTION

Denote by B^{N+1} a unique ball in R^{N+1} , surface of this ball denote by S^N :

$$S^N = \{x = (x_1, x_2, ..., X_{N+1}) \in R^{N+1} : \sum_{n=1}^{N+1} x_n^2 = 1\}$$

Let x and y arbitrary points in S^N . By $\gamma = \gamma(x, y)$ denote spherical distance between these two points. In fact γ is an angle between vectors x and y . It is clear that $\gamma \leq \pi$. By $B(x, r)$ denote a ball on a sphere S^N , with radius r and with the center at a point x :

$$B(x, r) = \{y \in S^N : \gamma(x, y) \leq r\}$$

❖ INTRODUCTION

Let Δ_s be Laplace-Beltrami operator on S^N. We have following way to calculate operator Δ_s , using Laplace's operator Δ in R^{N+1} (see for instance in [6].): let $f(x)$ a function determined on S^N ; extend it to R^{N+1}, by putting $\hat{f}(x) = f(\frac{x}{|x|})$, $x \in R^{N+1}$. Then $\Delta_s f = \Delta \hat{f}\big|_{S^N}$. Another way of determination of Δ_s is to represent Laplace operator Δ in R^{N+1} by spherical coordinates. In this case it would be easy to "separate" operator Δ_s by separation angled coordinates:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{N}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_s,$$

where operator Δ_s can be written in spherical coordinates $(\xi_1, \xi_2, ..., \xi_{N-1}, \zeta)$ as:

❖ INTRODUCTION

$$\Delta_s = \frac{1}{\sin^{N-1}\xi_1}\frac{\partial}{\partial\xi_1}\left(\sin^{N-1}\xi_1\frac{\partial}{\partial\xi_1}\right)+\frac{1}{\sin^2\xi_1\sin^{N-2}\xi_2}\frac{\partial}{\partial\xi_2}\left(\sin^{N-2}\xi_2\frac{\partial}{\partial\xi_2}\right)+\ldots.+$$

$$+\frac{1}{\sin^2\xi_1\sin^2\xi_2\ \ldots.\sin^2\xi_{N-1}}\frac{\partial^2}{\partial\zeta^2}.$$

❖ INTRODUCTION

Operator $-\Delta_s$ as a formal differential operator with domain of definition $C^\infty(S^N)$ is a symmetric, non negative and its closure $\overline{-\Delta_s}$ is a selfadjoint operator in $L_2(S^N)$. Eigenfunctions Y^k of the operator $-\Delta_s$, are called spherical harmonics. Spherical harmonics of a degree k and ℓ , $k \neq \ell$ are orthogonal . Corresponding eigenvalues are $\lambda_k = k(k+N-1)$, where $k = 0, 1, 2, ...$, and with frequency a_k equal to the dimension of the space of homogeneous harmonic polynomials of a degree k: $a_k = N_k - N_{k-2}$, where $N_k = \frac{(N+k)!}{N!k!}$. That is why for each k there are a_k number of spherical harmonics $\{Y_j^k\}\Big|_{j=1}^{a_k}$ corresponding to eigenvalue λ_k . A family of functions $\{Y_j^k\}\Big|_{j=1}^{a_k}$ is an orthonormal basis in the space of spherical harmonics of a degree k which we denote by $\aleph_k$.

❖ METHODOLOGY
FOURIER LAPLACE SERIES

Note that an arbitrary function $f \in L_2(S^N)$ can be represented in a unique way as Fourier series by spherical harmonics $\{Y_j^k\}\Big|_{j=1}^{a_k}$. Such a series is called Fourier-Laplace series on sphere:

$$f(x) = \sum_{k=0}^{\infty} \sum_{j=1}^{a_k} f_{k,j} Y_j^k(x), \quad (1.1)$$

where $f_{k,j} = \int_{S^N} f(y) Y_j^k(y) d\sigma(y)$, and equality (1.1) should be understanding in sense of $L_2(S^N)$.

❖ METHODOLOGY
FOURIER LAPLACE SERIES

Let denote by $S_n f(x)$ a partial sum of series (1.1). It is clear that in $S_n f(x)$ by changing order of integration and summation one can easily rewrite it as:

$$S_n f(x) = \int_{S^N} f(y)\Theta(x,y,n)d\sigma(y),$$

where a function $\Theta(x,y,n)$ is a spectral function (see in [1]) of a selfadjoint operator $\overline{-\Delta}$ and has a form:

$$\Theta(x,y,n) = \sum_{k=0}^{n}\sum_{j=1}^{a_k} Y_j^k(x)Y_j^k(y), \qquad (1.2)$$

and $S_n f(x)$ is called a spectral expansion of an element f correspondin to the operator $\overline{-\Delta}$ (see in [1]).

❖ METHODOLOGY

CHEZARO MEANS

Determine Chezaro means of order α of partial sums of series (1.1) by equality

$$S_n^\alpha f(x) = \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^\alpha \sum_{j=1}^{a_k} f_{k,j} Y_j^k(x), \tag{2.1}$$

where $A_n^\alpha = \frac{\Gamma(\alpha+m+1)}{\Gamma(\alpha+1)m!}$.

Definition 2.1. Series (1.1) is sumable to $f(x)$ by Chezaro means of order α if it is true that

$$\lim_{n\to\infty} S_n^\alpha f(x) = f(x) \tag{2.2}$$

SPHERICAL SYSTEM

If $(\rho, \theta_1, \theta_2, \ldots, \theta_{k-2}, \varphi)$ are the spherical coordinates of the point $x(x_1, x_2, \ldots, x_k)$, then

$$x_1 = \rho \cos\theta_1;$$
$$x_2 = \rho \sin\theta_1 \cos\theta_2;$$
$$x_3 = \rho \sin\theta_1 \sin\theta_2 \cos\theta_3;$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$x_{k-1} = \rho \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{k-2} \cos\varphi;$$
$$x_k = \rho \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{k-2} \sin\varphi;$$
$$0 \le \rho < \infty; \quad 0 \le \theta_i \le \pi \quad (i = \overline{1, k-2}), \quad 0 \le \varphi \le 2\pi,$$

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SPHERICAL SYSTEM

If for $k = 3$, (x, y, z) are the Cartesian coordinates of the point M, and (ρ, θ, φ) are the spherical coordinates, then the equalities (1.1),(1.3) and (1.4) take the form

$$x = \rho \sin\theta \cos\varphi, \quad y = \rho \sin\theta \sin\varphi, \quad z = \rho \cos\theta,$$

$$dS_\rho^2 = \rho^2 \cdot \frac{\partial}{\partial\rho}\left(\rho^2 \frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2} D_3,$$

$$D_3 = \frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\left(sin\theta \frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta} \cdot \frac{\partial^2}{\partial\varphi^2}.$$

FUNCTIONAL SPACES

$L_p(S^{k-1})$, $1 \le p < \infty$ $(L_1(S^{k-1}) = L(S^{k-1}))$ is the space of the function f with the norm

$$\|f\|_{L_P(S^{k-1})} = \left(\int_{S^{k-1}} |f(x)|^P dS^{k-1}(x) \right)^{1/P}, \quad 1 \le P < \infty.$$

$$dS_\rho^{k-1}(x) = \rho^{k-1} \sin^{k-2}\theta_1 \ldots \sin\theta_{k-2} d\theta_1 \ldots d\theta_{k-2} d\varphi$$

For $P = \infty$ it is assumed that the space $L_\infty(S^{k-1}) = C(S^{k-1})$ consists of continuous functions with the norm

$$\|f\|_{C(S^{k-1})} = \max_{x \in S^{k-1}} |f(x)|.$$

❖ METHODOLOGY
FUNCTIONAL SPACES

For any positive number a define an operator $\left(I + \overline{-\Delta_s}\right)^{\frac{a}{2}}$ with domain $C^{\infty}(S^N)$ as follows

$$\left(I + \overline{-\Delta_s}\right)^{\frac{a}{2}} f(x) = \sum_{k=0}^{\infty} \left(1 + k(k+N-1)\right)^{\frac{a}{2}} \sum_{j=1}^{a_k} f_{k,j} Y_j^k(x),$$

where I is the unit operator.

Now for the positive number a and $1 \le p \le \infty$ we denote by $W_p^a(S^N)$ a closure of the space $C^{\infty}(S^N)$ with respect a norm $\|f\|_{p,a} = \|\left(I + \overline{-\Delta_s}\right)^{\frac{a}{2}} f\|_p$. It is not difficult to check that this norm is equivalent to the norm $\|f\|_p + \|\left(\overline{-\Delta_s}\right)^{\frac{a}{2}} f\|_p$. The space $W_p^a(S^N)$ is a complete normed spaces (the Banach space) and is called the space of the Bessels potentials on a unit sphere S^N.

❖ METHODOLOGY

FUNCTIONAL SPACES

For the function f defined on a unite sphere S^N we define a shifting operator as follows

$$(S_h f)(x) = \frac{1}{|S^{N-1}| sin^{N-1} h} \int_{x \cdot y = cosh} f(y) dl_h(y),$$

where $|S^{N-1}|$ is a surface area of a $N-1$ dimensional unite sphere, x and y from S^N, $dl_h(y)$ is an element of the volume of the $N-1$ dimensional sphere $x \cdot y = cosh$ with the radius equal $sin^{N-1} h$. Note that the shifting operator S_h is a bounded operator in $L_p(S^N)$.

By using the operator S_h we define an averaged spherical differences of order k as follows

$$\Delta_h^k f(x) = \sum_{j=0}^{k} (-1)^{k+j} C_h^k (S_{kh} f)(x).$$

Denote by $H_p^a(S^N)$ a space of function satisfying the condition with respect a norm

$$\|f\|_{H_p^a} = \|f\|_p + sup_h\, h^{-a} \|\Delta_h^k f(x)\|_p < \infty,$$

where k is a positive integer number grater than a. The space $H_p^a(S^N)$ is called the Nikol'skii space (class) of functions on a sphere S^N.

❖ MAIN RESULTS

THEOREM 3.1. *Let* Ω *is a domain on* S^N *and let* $f \in \varepsilon'(S^N) \cap C(\Omega) \cap H_p^{-a}(S^N)$. *If*

$$1 \le p \le \infty,\ a > 0,\ \alpha > -1,\ \alpha > max\{\frac{N}{p} - 1, \frac{N-1}{2}\} + a,$$

then

$$\lim_{n \to \infty} S_n^\alpha f(x) = f(x)$$

uniformly on each compact set $K \subset \Omega$.

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THANK YOU!!!